Unconditional bases and Daugavet renormings

Joint work with Rainis Haller, Johann Langemets and Yoël Perreau

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1. R. Haller, J. Langemets, Y. Perreau, and T. Veeorg, *Unconditional bases and Daugavet renormings* (2023). Preprint, arXiv:2303.07037

Preliminaries

We consider only real Banach spaces and use common notation.
Let X be a Banach space. We will denote the closed unit ball by B_X, the unit sphere by S_X and the dual space by X^*.

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A **slice** of the unit ball B_X is a set

$$S(x^*, \alpha) = \{y \in B_X : x^*(y) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$.
A point $x \in B_X$ is called a **denting point** of B_X if it is contained in slices of B_X of arbitrarily small diameter.

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Radon–Nikodým property and Daugavet property

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A Banach space X has the **Daugavet property**, if for every $x \in S_X$, for every $\varepsilon > 0$ and for every slice S of B_X there exists $y \in S$ such that $\|x - y\| \geqslant 2 - \varepsilon$.

Daugavet points and Δ-points

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Let X be a Banach space, and let $x \in S_X$. We say that x is

1. a *Daugavet point* if $\sup_{y \in S} \|x - y\| = 2$ for every slice S of B_X;

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3. a *∇-point* if $\sup_{y \in S} \|x - y\| = 2$ for every slice S of B_X that does not contain the element x.

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    - Yes, there exists a Lipschitz-free space that has the Radon–Nikodým property and a Daugavet point (V. 2023).

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4. Can every infinite dimensional Banach space be renormed to have a ∇-point?

Renormings with ∇-points

Proposition (Haller, Langemets, Perreau, V. 2023)

Let X and Y be Banach spaces, and let $x \in S_X$. Then x is a ∇-point in X if and only if $(x, 0)$ is a ∇-point in $X \oplus_1 Y$.

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Proposition (Haller, Langemets, Perreau, V. 2023)

Every Banach space can be renormed with a ∇-point.

Renormings with Daugavet points

Theorem (Haller, Langemets, Perreau, V. 2023)

Let X be an infinite dimensional Banach space with unconditional weakly null Schauder basis. Then X can be renormed with a Daugavet point.

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Sketch of proof.

Renorm X with 1-unconditional basis (e_n). Let $Y := \overline{\mathrm{span}}\{e_n\}_{n\geqslant 2}$ and let A be the set of all finitely supported elements in the positive cone of Y.

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$$B_{(X, |||\cdot|||)} := \overline{\text{conv}}\{\pm(e_1 + 2x) \colon x \in A \cap B_X\}.$$

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Then for every $x \in A \cap S_X$ we have

$$|||e_1 - (e_1 + 2x)||| = 2\,|||x||| = 2 \text{ and } |||e_1 + (e_1 + 2x)||| = 2\,|||e_1 + x||| = 2,$$

and thus e_1 is a Daugavet point. □

Open question

1. Can every infinite dimensional Banach space be renormed to have a Daugavet point?
   Yes, if
   - X has unconditional weakly null basis;
   - X is ℓ_1;
   - X contains a complemented subspace Y that can be renormed with a Daugavet point.