Some recent progresses in renorming theory

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(Joint work with C. A. De Bernardi and A. Preti)

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In these cases we say that $(X, |\cdot|)$ is a renorming of $(X, \|\cdot\|)$.

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Goal: Given an infinite-dimensional Banach space $(X, \|\cdot\|)$, to find a renorming $(X, |\cdot|)$ with nice rotund/smooth properties.

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- ℓ_∞ does not admit a LUR renorming[Lindenstrauss-Troyanski, 1971-1972].

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On the other hand if Γ is an uncountable set, then $\ell_\infty(\Gamma)$ does not admit a strictly convex renorming [Day,1955].

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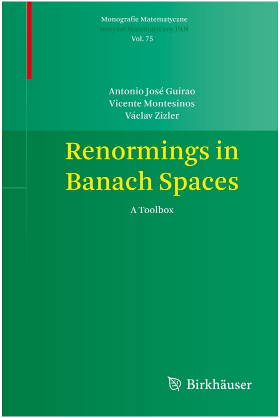
- Every separable Banach space with a separable dual has a LUR, Fréchet renorming.

Problem



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Renormings in Banach Spaces
A Toolbox
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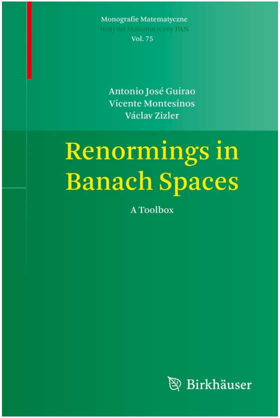

Problem 52.1.1.5: Does every infinite-dimensional separable space admit a norm that is rotund and Gâteaux smooth but not LUR?

Main result

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Theorem (De Bernardi, S.)

Every infinite-dimensional separable Banach space admits an average locally uniformly rotund (and hence rotund) Gâteaux smooth equivalent norm $|\cdot|$ which is not locally uniformly rotund.

Construction in ℓ_2

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is a closed, convex, symmetric subset of ℓ_2 with nonempty interior.

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What about the general case?

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$$D := \mathrm{conv}(T[B_{\ell_2}] \cup B_{(X,\|\!|\cdot|\!\|)}) \qquad |x|^2 = \|x\|^2 + \sum_{n=2}^{\infty} 2^{-n} f_n(x)^2.$$

Where $f_n = g_n/\|\!|g_n|\!\|^*$ $(n \in \mathbb{N})$.

Rotund not MLUR

Problem 52.3.3 [GMZ]: Can every infinite-dimensional separable Banach space be renormed to be Rotund and not MLUR?

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Definition

A norm $\|\cdot\|$ is called midpoint locally uniformly rotund (MLUR) if whenever $x \in S_X$ and $x_n \in X$ are such that $\|x + x_n\| \to 1$ and $\|x - x_n\| \to 1$, then $x_n \to 0$.

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A norm $\|\cdot\|$ is called weakly uniformly rotund (WUR) if whenever $x_n - y_n \to 0$ in the weak topology of X whenever $x_n, y_n \in S_X$ are such that $\|x_n + y_n\| \to 2$.

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Both questions have positive answer (Draga, 2015).

A smooth generalization

Theorem (De Bernardi, Preti, S.)

Every infinite-dimensional separable Banach space admits an equivalent norm which is Rotund, Gâteaux smooth and not MLUR.

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Every infinite-dimensional separable Banach space admits an equivalent norm which is Rotund, Gâteaux smooth and not MLUR.

Theorem (De Bernardi, Preti, S.)

Every infinite-dimensional Banach space with separable dual admits an equivalent norm which is WUR Fréchet smooth and not MLUR.

Solved problems

Problem	Answer	Paper
52.1.1.5	Y	[DS]
52.1.2.1	Y	[Q]
52.1.2.4	Y	[Q]
52.1.3.3	Y	[Q]
52.1.4.2	Y	[Q]
52.1.4.6	Y	[HQ]
52.3.1	Y	[HQ]
52.3.3	Y	[D]
52.3.4	Y	[HQ]
52.3.6	Y	[D]
52.3.7	Y	[HQ]

Reference

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- S. Draga, *On weakly locally uniformly rotund norms which are not locally uniformly rotund,* Coll. Math. 138 (2015) 241–246
- P. Hájek, A. Quilis, *Counterexamples in rotundity of norms in Banach spaces,* arXiv:2302.11041
- A. Quilis, *Renormings preserving local geometry at countably many points in spheres of Banach spaces and applications,* J. Math. Anal. Appl. 526 (2023) 127276

Thank you for your attention!