Continuous operators from spaces of Lipschitz functions

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Joint work in progress with C. Bargetz and J. Kąkol.

First Motivation

Long tradition of intensive studies on the existence of surjective operators

$$(C(X), \tau) \longrightarrow (C(Y), \sigma),$$

where X, Y are Tychonoff spaces (usually compact, sometimes metric) and τ, σ are linear topologies (usually: sup norm topology, weak topology, compact-open topology, pointwise topology).

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What about spaces $\mathrm{Lip}_0(M)$?

Let M and N be metric spaces. What can we say about the existence of surjective operators

$$(E, \tau) \longrightarrow (F, \sigma),$$

where $E \in \{C(M), \mathrm{Lip}_0(M)\}$ and $F \in \{C(N), \mathrm{Lip}_0(N)\}$?

The situation is pesimistic...

Obvious observation

For a metric space M, the identity operators $\mathrm{Lip}_0(M) \longrightarrow \mathrm{Lip}_0(M)_w$ and $\mathrm{Lip}_0(M)_w \longrightarrow \mathrm{Lip}_0(M)_p$ are continuous linear surjections.

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Theorem, 1st part

Let M and N be infinite metric spaces. Then, there is no continuous surjection $T\colon X \to Y$ for the following pairs of spaces X and Y:

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2. $X = \mathrm{Lip}_0(M)_p$ and $Y = \mathrm{Lip}_0(N)$,
3. $X = \mathrm{Lip}_0(M)_p$ and $Y = C_p(N)$.

$\mathrm{Lip}_0(M)$ contains ℓ_∞

Theorem (Hájek–Novotný)

For every metric space M, the space $\mathcal{F}(M)$ contains a complemented copy of $\ell_1(d(M))$.

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Consequences

Let M and N be infinite metric spaces.

1. $d(\mathrm{Lip}_0(M)) = 2^{d(M)}$

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Theorem

For every metric space M, we have:
$d(\mathrm{Lip}(M)_p) = d(C_p(M)) \leqslant d(M)$.

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Theorem (Rosenthal, Lacey)

For every infinite compact space K, the space $C(K)$ admits a (continuous linear) operator onto c_0 or ℓ_2.

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Question

What about operators from $\mathrm{Lip}_0(M)$ onto c_0?

The case of $C(K)$-spaces

Theorem (Räbiger, Schachermayer, Cembranos, Diestel,...)

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Let X be a Banach space. If the dual X^* contains a copy of c_0, then this copy is not complemented.

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Theorem

Let X be a Banach space. If the dual X^* contains a copy of c_0, then this copy is not complemented. Consequently, there is no complemented copy of c_0 in $\mathrm{Lip}_0(M)$ for any metric space M.

$\mathrm{Lip}_0(c_0)$ and $\mathrm{Lip}_0(\ell_1)$

Theorem (Lindenstrauss)

For every Banach space X, the space $\mathrm{Lip}_0(X)$ contains a complemented copy of the dual space X^*.

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For every Banach space X, the space $\mathrm{Lip}_0(X)$ contains a complemented copy of the dual space X^*.

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$\mathrm{Lip}_0(c_0)$ contains a complemented copy of ℓ_1, so it admits an operator onto c_0.

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Theorem (Dalet)

$\mathrm{Lip}_0(\ell_1)$ contains a complemented copy of ℓ_1.

Lipschitz retracts

Proposition

Let $M \subseteq N$. If M is a Lipschitz retract of N, then $\mathrm{Lip}_0(M)$ is isomorphic to a complemented subspace of $\mathrm{Lip}_0(N)$.

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c_0 is an absolute Lipschitz retract, i.e. for every metric spaces X and Y, if $c_0 \subseteq X$ and $\varphi\colon c_0 \to Y$ is Lipschitz, then φ extends to a Lipschitz mapping $\Phi\colon X \to Y$.

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5. $X = \mathcal{L}(Y, Z)$ for a normed space Y and a Banach space Z which contains c_0,
6. $X = Bil(Y \times W, Z)$ for some Banach spaces Y, W, Z such that Z contains c_0 (recall: $Bil(Y \times W, Z) \cong \mathcal{L}(Y \widehat{\otimes}_\pi W, Z)$),

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7. $X = \mathrm{Lip}_0(M)$ for some metric space M,
8. $X = \mathcal{F}(M)$ for some metric space M.

“Guilty” structures

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If a metric space M contains a bilipschitz copy of the unit sphere S_{c_0} of c_0, then $\mathrm{Lip}_0(M)$ admits an operator onto c_0.

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Theorem (Raynaud)

ℓ_1 does not contain any bilipschitz (or even uniform) copies of S_{c_0}.

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Another example

$M = \bigsqcup_{n\in\mathbb{N}} \ell_\infty^n$

$\mathrm{Lip}_0(M) \simeq \left(\bigoplus_{n=0}^{\infty} \mathrm{Lip}_0(\ell_\infty^n)\right)_\infty \simeq \mathrm{Lip}_0(c_0)$

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Question

Does $\mathrm{Lip}_0([0,1]^2)$ admit an operator onto c_0?

Recall: $\mathrm{Lip}_0([0,1]) \simeq \ell_\infty$.

$\mathrm{Lip}_0(\ell_2)$

Question

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If not, then:

For every infinite-dimensional Banach space X of density $\kappa \geqslant \aleph_0$, $\mathrm{Lip}_0(X)$ contains a complemented copy of the dual space Y^* of every Banach space Y of density $\leqslant \kappa$.

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Johnson (1971): There exists a separable Banach space E such that E^* contains a complemented copy of F^* for every separable F.

The end

Thank you for your attention!