Lipschitz-free spaces and representing measures

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Abstract

Let M be a complete metric space with base point 0, and let $\mathrm{Lip}_0(M)$ denote the Banach space of Lipschitz functions on M that vanish at 0. The **Lipschitz-free** space $\mathcal{F}(M) \subseteq \mathrm{Lip}_0(M)^*$ over M is defined as the closed linear span of the set of functionals $f \mapsto f(x)$, $x \in M$, and is an isometric predual of $\mathrm{Lip}_0(M)$. These spaces have important applications in the linear and nonlinear theory of Banach spaces.

Let $\widetilde{M} = \{(x,y) \in M \times M : x \neq y\}$ and denote by $\beta\widetilde{M}$ its Stone-Čech compactification. Define the **de Leeuw transform** $\Phi : \mathrm{Lip}_0(M) \to C(\beta\widetilde{M})$ by

$$\Phi f(x,y) = \frac{f(x) - f(y)}{d(x,y)}, \quad f \in \mathrm{Lip}_0(M),\ (x,y) \in \widetilde{M},$$

and extending continuously to $\beta\widetilde{M}$. This is a linear isometric embedding whose dual $\Phi^* : C(\beta\widetilde{M})^* \to \mathrm{Lip}_0(M)^*$ is therefore a quotient map. Thus, every element of $\mathrm{Lip}_0(M)^*$ can be represented (non-uniquely) by Radon measures on $\beta\widetilde{M}$ via the de Leeuw transform.

Given $\psi \in \mathrm{Lip}_0(M)^*$, there exists a positive Radon measure μ on $\beta\widetilde{M}$ such that $\Phi^*\mu = \psi$ and $\|\mu\| = \|\psi\|$. We call such a measure μ an **optimal representation** of ψ. In this mini-course, we investigate the properties of optimal representations μ such that $\mu(\widetilde{M}) = \|\mu\|$ (in which case $\Phi^*\mu \in \mathcal{F}(M)$) or more generally $\mu(\widetilde{M}) > 0$. We illustrate connections with cyclical monotonicity from optimal transport theory, elements of $\mathcal{F}(M)$ that can be induced by measures on M, and present an application of this work to the extreme point problem in Lipschitz-free spaces.

This is joint work with Ramón Aliaga (Universitat Politècnica de València) and Eva Pernecká (Czech Technical University, Prague).

1 Background and motivation

1.1 Lipschitz-free spaces and convex series of molecules

Definition 1.1

1. Let (M, d) be a complete metric space with base point 0. Define the Banach space

$$\mathrm{Lip}_0(M) = \{f : M \to \mathbb{R} \ : \ f \text{ is Lipschitz and } f(0) = 0\},$$

with norm

$$\|f\| := \mathrm{Lip}(f) \ = \ \sup\left\{\frac{f(x) - f(y)}{d(x,y)} \ : \ x, y \in M,\ x \neq y\right\}.$$

2. Define $\widetilde{M} = \left\{(x,y) \in M^2 \ : \ x \neq y\right\}$ and the set $E = \left\{m_{xy} \ : \ (x,y) \in \widetilde{M}\right\} \subseteq S_{\mathrm{Lip}_0(M)^*}$ of **elementary molecules** m_{xy}, where

$$\left\langle m_{xy}, f\right\rangle \ = \ \frac{f(x) - f(y)}{d(x,y)}, \qquad f \in \mathrm{Lip}_0(M).$$

3. Define the **(Lipschitz-) free** Banach space

$$\mathcal{F}(M) = \overline{\mathrm{span}}^{\|\cdot\|}(E) \subseteq \mathrm{Lip}_0(M)^*.$$

The free space $\mathcal{F}(M)$ is an isometric predual of $\mathrm{Lip}_0(M)$. Free spaces, known also as Arens-Eells spaces and transportation cost spaces, are a popular field of study and have found numerous applications in the linear and non-linear theory of Banach spaces. They give us a tool to linearise Lipschitz maps in a canonical way. We refer the reader to [4, Introduction] and references therein for an overview of this topic. A comprehensive introduction to Lipschitz and Lipschitz-free spaces (and much more) can be found in [9], where the latter are known as Arens-Eells spaces. In these lectures we present some recent work from [4], which is the first in a planned series of papers on so-called de Leeuw representations of elements of $\mathrm{Lip}_0(M)^*$ (e.g. [5]). The first result is well-known.

Proposition 1.2 *We have* $B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}^{\|\cdot\|}(E)$.

Proof. Evidently $\overline{\mathrm{conv}}^{\|\cdot\|}(E) \subseteq B_{\mathcal{F}(M)}$. Conversely, given $m \notin \overline{\mathrm{conv}}^{\|\cdot\|}(E)$, by the Hahn-Banach separation theorem there exists $f \in \mathrm{Lip}_0(M)$ such that

$$\langle m, f\rangle > \sup\left\{\langle p, f\rangle \ : \ p \in \overline{\mathrm{conv}}^{\|\cdot\|}(E)\right\} = \|f\|,$$

hence $\|m\| > 1$ and $m \notin B_{\mathcal{F}(M)}$. □

Exercise 1.3 Let X be a normed space and let $H \subseteq X$ with $B_X = \overline{\mathrm{conv}}^{\|\cdot\|}(H)$. Show that, given $x \in X$ and $\varepsilon > 0$, there exist $x_n \in H$ and $a_n \geq 0$, $n \in \mathbb{N}$ satisfying

$$x = \sum_{n=1}^{\infty} a_n x_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n \leq \|x\| + \varepsilon.$$

This exercise is very similar to [8, Lemma 3.100]. As a corollary we obtain the following well-known result.

Corollary 1.4 *Given $m \in \mathcal{F}(M)$ and $\varepsilon > 0$, there exist $(x_n, y_n) \in \widetilde{M}$ and $a_n \geq 0$, $n \in \mathbb{N}$, such that*

$$m = \sum_{n=1}^{\infty} a_n m_{x_n y_n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n \leq \|m\| + \varepsilon.$$

We distinguish those elements of $\mathcal{F}(M)$ for which ε above can be set to 0.

Definition 1.5 (Aliaga, Rueda Zoca 20 [6]) We say that $m \in \mathcal{F}(M)$ is a **convex series of molecules** if there exist $(x_n, y_n) \in \widetilde{M}$ and $a_n \geq 0$, $n \in \mathbb{N}$, such that

$$m = \sum_{n=1}^{\infty} a_n m_{x_n, y_n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n = \|m\|.$$

1.2 The extreme point conjecture

Identifying the set $\operatorname{ext} B_X$ extreme points (if any) of the unit ball of a Banach space X is a standard goal when trying to understand its structure. The question of characterising $\operatorname{ext} B_{\mathcal{F}(M)}$ dates back to results of Weaver in the 1990s (see e.g. [9, Sections 3.5 and 3.6]). Weaver has posed the following conjecture.

Conjecture 1.6 *Every extreme point of $B_{\mathcal{F}(M)}$ is an elementary molecule:* $\operatorname{ext} B_{\mathcal{F}(M)} \subseteq E$.

This conjecture is natural, given e.g. Proposition 1.2. A number of partial positive results have been obtained over the years; in these lectures we present two of them.

Proposition 1.7 (APPP 20 [3, Remark 3.4]) *If $m \in \operatorname{ext} B_{\mathcal{F}(M)}$ is a convex series of molecules then $m \in E$.*

Proof. Let $m = \sum_{n=1}^{\infty} a_n m_{x_n y_n} \in \operatorname{ext} B_{\mathcal{F}(M)}$ be a convex series of molecules where, without loss of generality, $a_1 > 0$. If $a_1 = 1$ then $m = m_{x_1 y_1}$. Else $a_1 \in (0, 1)$ and,

$$m = a_1 m_{x_1 y_1} + (1 - a_1) \sum_{n=2}^{\infty} \frac{a_n}{1 - a_1} m_{x_n y_n}.$$

giving $m = m_{x_1 y_1}$. □

Definition 1.8 The metric space M is **proper** if all of its closed bounded subsets are compact.

Every compact metric space is proper, and every closed subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, (with respect to any norm) is proper. No proper space can contain a bounded uniformly separated infinite sequence, hence no unit ball B_X, $\dim X = \infty$, is proper.

Theorem 1.9 (Aliaga 22 [1, Theorem 1]) *If M is proper then* $\operatorname{ext} B_{\mathcal{F}(M)} \subseteq E$.

In these lectures we generalise the notion of convex series of molecules in order to obtain a new positive case of Conjecture 1.6.

2 de Leeuw representations and convex integrals of molecules

2.1 Optimal de Leeuw representations

To generalise the notion of convex series of molecules, we use a representation of elements $\mathrm{Lip}_0(M)^*$ due to de Leeuw.

Definition 2.1 Given a Tychonoff (completely regular Hausdorff) topological space X, the **Stone-Čech compactification** βX of X is a Hausdorff compactification of X characterised (up to homeomorphism) by the **extension property**: any continuous map $f : X \to K$ (K compact Hausdorff) can be extended uniquely to a continuous map $f : \beta X \to K$.

If X is completely metrisable then X is a G_δ subset of βX.

Definition 2.2 Define $\Phi : \mathrm{Lip}_0(M) \to C(\beta\widetilde{M})$ by first setting

$$(\Phi f)(x,y) = \frac{f(x) - f(y)}{d(x,y)} = \left\langle m_{xy}, f \right\rangle, \qquad (x,y) \in \widetilde{M},$$

and then extending continuously to its Stone-Čech compactification $\beta\widetilde{M}$. We call both Φ and its dual $\Phi^* : C(\beta\widetilde{M})^* \equiv \mathcal{M}(\beta\widetilde{M}) \to \mathrm{Lip}_0(M)^*$ **de Leeuw transforms**.

Exercise 2.3 Show that Φ is an isometric embedding and Φ^* is a quotient map.

Definition 2.4 Given $\psi \in \mathrm{Lip}_0(M)^*$, we call $\mu \in \mathcal{M}(\beta\widetilde{M})$ a **(de Leeuw) representation** of ψ if $\Phi^*\mu = \psi$.

We have $\|\Phi^*\mu\| \leq \|\mu\|$ always; we focus on those **positive** μ for which $\|\Phi^*\mu\| = \|\mu\|$.

Definition 2.5 Define the set of **optimal representations**

$$\mathcal{M}_{\mathrm{op}}(\beta\widetilde{M}) = \left\{\mu \in \mathcal{M}(\beta\widetilde{M}) \ : \ \mu \geq 0 \text{ and } \|\Phi^*\mu\| = \|\mu\|\right\}.$$

Example 2.6 Let $(x,y) \in \widetilde{M}$. Then $\Phi^*\delta_{(x,y)} = m_{xy}$ and $\delta_{(x,y)} \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ as $\left\|\delta_{(x,y)}\right\| = 1 = \left\|m_{xy}\right\|$.

Example 2.7 Let $M := [0,1]$ have base point 0. Define positive $\mu_n \in \mathcal{M}(\beta\widetilde{M})^+$, $n \geq 0$, by

$$\mu_n = \frac{1}{2^n} \sum_{i=1}^{2^n} \delta_{\left(\frac{i}{2^n}, \frac{i-1}{2^n}\right)}.$$

Then $\mu_n \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$, with $\Phi^*\mu_n = m_{10}$: $\|\mu_n\| = 1 = \|m_{10}\|$ and

$$\langle \Phi^*\mu_n, f\rangle = \langle \mu_n, \Phi f\rangle = \frac{1}{2^n} \sum_{i=1}^{2^n} \frac{f\left(\frac{i}{2^n}\right) - f\left(\frac{i-1}{2^n}\right)}{2^{-n}} = f(1) - f(0) = \langle m_{10}, f\rangle, \qquad f \in \mathrm{Lip}_0(M),$$

giving $\Phi^*\mu_n = m_{10}$. Any w^*-accumulation point μ of (μ_n) is also an optimal representation of m_{10}; any such point is supported entirely on the 'diagonal' $\beta\widetilde{M} \setminus \widetilde{M}$. On the other hand, the representation $\mu' := \delta_{(1,0)} + \delta_{(\frac{1}{2},0)} + \delta_{(0,\frac{1}{2})}$ is not optimal: $\|\mu'\| = 3$, with the mass reflected in the diagonal 'cancelling out' because $\Phi f(x,y) = -\Phi f(y,x)$, $(x,y) \in \widetilde{M}$.

We explore some basic properties of $\mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$. The reflection above happens to be key.

Definition 2.8 Define the **reflection map** $\mathfrak{r}:\beta\widetilde{M}\to\beta\widetilde{M}$ by setting $r(x,y)=(y,x)$, $(x,y)\in\widetilde{M}$, and then extending continuously to $\beta\widetilde{M}$. Then define the isometry $R: C(\beta\widetilde{M})\to C(\beta\widetilde{M})$ by $Rf=f\circ\mathfrak{r}$.

Note that $R^*\mu=\mathfrak{r}_\#\mu$, where # denotes pushforward.

Exercise 2.9 Show that $R\Phi=-\Phi$, and thus $\Phi^*R^*=-\Phi^*$.

We need the next definition before presenting our list of basic properties of $\mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$.

Definition 2.10 Let $\mu\in\mathcal{M}(\beta\widetilde{M})$ and let $E\subseteq\beta\widetilde{M}$ be Borel. We say that μ is **concentrated** on E if $\mu(A)=\mu(A\cap E)$ whenever $A\subseteq\beta\widetilde{M}$ is Borel. We define the **restriction** $\mu\restriction_E$ of μ to E by $\mu\restriction_E(A)=\mu(A\cap E)$, A Borel.

Of course, $\mu\restriction_E$ is concentrated on E.

Proposition 2.11

1. *For any* $\psi\in\mathrm{Lip}_0(M)^*$ *there is* $\mu\in\mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ *such that* $\Phi^*\mu=\psi$.
2. *If* $\mu\in\mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ *then* $c\cdot\mu\in\mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ *for every* $c\geq 0$.
3. *If* $\mu\in\mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ *and* $\lambda\in\mathcal{M}(\beta\widetilde{M})$ *satisfies* $0\leq\lambda\leq\mu$, *then* $\lambda\in\mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$.
4. *If* $\mu\in\mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ *and* $E\subseteq\beta\widetilde{M}$ *is Borel then* $\mu\restriction_E\in\mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$.

Proof. (1) (is [1, Proposition 3] with a different proof) Let $\psi\in\mathrm{Lip}_0(M)^*$. Then $\psi\circ\Phi^{-1}\in(\Phi\,\mathrm{Lip}_0(M))^*$. By the Hahn-Banach theorem we extend $\psi\circ\Phi^{-1}$ to $\nu\in\mathcal{M}(\beta\widetilde{M})$, such that $\|\nu\|=\left\|\psi\circ\Phi^{-1}\right\|=\|\psi\|$. Then

$$\langle\Phi^*\nu,f\rangle=\langle\nu,\Phi f\rangle=\langle\psi,f\rangle\,,$$

giving $\Phi^*\nu=\psi$. Now set $\mu=\nu^++R^*\nu^-$, where $\nu=\nu^+-\nu^-$ is the Jordan decomposition of ν. Then

$$\|\mu\|=\left\|\nu^+\right\|+\left\|R^*\nu^-\right\|=\left\|\nu^+\right\|+\left\|\nu^-\right\|=\|\nu\|\,,$$

and, by Exercise 2.9

$$\Phi^*\mu=\Phi^*\nu^++\Phi^*R^*\nu^-=\Phi^*(\nu^+-\nu^-)=\Phi^*\nu=\psi.$$

(2) is trivial. For part (3) notice that

$$\|\Phi^*\mu\|\leq\|\Phi^*\lambda\|+\|\Phi^*(\mu-\lambda)\|\leq\|\lambda\|+\|\mu-\lambda\|=\|\mu\|=\|\Phi^*\mu\|$$

and therefore $\|\lambda\|=\|\Phi^*\lambda\|$ as well. (4) is a particular case of (3). □

2.2 Convex integrals of molecules

Recall that $\widetilde{M}$ is a G_δ (hence Borel) subset of $\beta\widetilde{M}$. Let $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$, with $\psi = \Phi^*\mu$. Then

$$\begin{aligned}\langle\psi, f\rangle = \langle\mu, \Phi f\rangle &= \int_{\beta\widetilde{M}} (\Phi f)(\zeta)\,\mathrm{d}\mu(\zeta) \\ &= \int_{\beta\widetilde{M}\setminus\widetilde{M}} (\Phi f)(\zeta)\,\mathrm{d}\mu(\zeta) + \int_{\widetilde{M}} (\Phi f)(x,y)\,\mathrm{d}\mu(x,y), \qquad f \in \mathrm{Lip}_0(M).\end{aligned}$$

Hereafter we will be interested in when μ can be chosen to be concentrated on $\widetilde{M}$, so that the first integral on the second line above vanishes. Before presenting the next key result, it will help to recall the notion of the **support** of a measure.

Definition 2.12 Let $\mu \in \mathcal{M}(\beta\widetilde{M})$ be positive. Define

$$\begin{aligned}\mathrm{supp}(\mu) &= \left\{\zeta \in \beta\widetilde{M} \;:\; \mu(U) > 0 \text{ whenever } U \ni \zeta \text{ is open in } \beta\widetilde{M}\right\} \\ \text{and }\ \mathrm{supp}_{\widetilde{M}}(\mu) &= \left\{(x,y) \in \widetilde{M} \;:\; \mu(U) > 0 \text{ whenever } U \ni (x,y) \text{ is open in } \widetilde{M}\right\}.\end{aligned}$$

For such μ it holds that μ is concentrated on $\mathrm{supp}(\mu)$: $\|\mu\| = \mu(\mathrm{supp}(\mu))$, and if μ is concentrated on $\widetilde{M}$ then $\mathrm{supp}_{\widetilde{M}}(\mu) = \mathrm{supp}(\mu) \cap \widetilde{M}$.

Proposition 2.13 *If $\mu \in \mathcal{M}(\beta\widetilde{M})$ then*

$$\Phi^*(\mu\restriction_{\widetilde{M}}) = \int_{\widetilde{M}} m_{xy}\,\mathrm{d}\mu(x,y)$$

as a Bochner integral on $\mathcal{F}(M)$.

Proof. Assume without loss of generality that $\mu = \mu\restriction_{\widetilde{M}}$. We first check that the integral in the statement is a valid Bochner integral. Since $\left\|m_{xy}\right\| = 1$ and $\|\mu\| < \infty$, it is enough to verify that the mapping $(x,y) \mapsto m_{xy}$ is measurable, i.e. that it is weakly measurable and almost separably valued (see e.g. [7, Propositions 5.1 and 5.2]). The former means that the mapping $(x,y) \mapsto \left\langle m_{xy}, f\right\rangle$ is measurable for each $f \in \mathcal{F}(M)^* = \mathrm{Lip}_0(M)$, which is obvious as that mapping is precisely Φf. For the latter, as μ is Radon, μ is concentrated on $\mathrm{supp}(|\mu|) \cap \widetilde{M} = \mathrm{supp}_{\widetilde{M}}(|\mu|)$. This set is separable as $\widetilde{M}$ is metrisable, so we conclude that the integral is valid and represents an element of $\mathcal{F}(M)$. To verify the equality, we only need to check that

$$\left\langle \int_{\widetilde{M}} m_{xy}\,\mathrm{d}\mu(x,y), f\right\rangle = \int_{\widetilde{M}} \left\langle m_{xy}, f\right\rangle \mathrm{d}\mu(x,y) = \int_{\widetilde{M}} \Phi f(x,y)\,\mathrm{d}\mu(x,y) = \langle \Phi^*\mu, f\rangle$$

for any $f \in \mathrm{Lip}_0(M)$. □

The converse of Proposition 2.13 does not hold, in the sense that not every de Leeuw representation μ of an element of $\mathcal{F}(M)$ is concentrated on $\widetilde{M}$; in fact, sometimes they must be supported entirely outside $\widetilde{M}$ – see Theorem 3.8 below.

Definition 2.14 We say that $m \in \mathcal{F}(M)$ is a **convex integral of molecules** if $m = \Phi^*\mu$ for some $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ concentrated on $\widetilde{M}$.

3 Results on convex integrals of molecules

3.1 Relationship with convex series of molecules

Proposition 3.1 *Every convex series of molecules is a convex integral of molecules.*

Proof. Let $m = \sum_{n=1}^{\infty} a_n m_{x_n y_n}$, where $a_n \geq 0$, $(x_n, y_n) \in \widetilde{M}$, $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} a_n = \|m\|$. Set $\mu = \sum_{n=1}^{\infty} a_n \delta_{(x_n,y_n)}$, which is concentrated on $\widetilde{M}$. Then $\|\mu\| = \sum_{n=1}^{\infty} a_n = \|m\|$ and

$$\Phi^* \mu = \sum_{n=1}^{\infty} a_n \Phi^* \delta_{(x_n,y_n)} = \sum_{n=1}^{\infty} a_n m_{x_n y_n} = m. \qquad \square$$

Proposition 3.2 *If M is scattered, then every convex integral of molecules in $\mathcal{F}(M)$ is also a convex series of molecules.*

Proof (sketch). Let $m \in \mathcal{F}(M)$ have a representation $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ concentrated on $\widetilde{M}$. As M is scattered, $\widetilde{M}$ is scattered also, and it follows that $\mathrm{supp}_{\widetilde{M}}(\mu)$ (which is a Polish space) is countable. Thus μ is atomic, i.e. $\mu = \sum_{n=1}^{\infty} a_n \delta_{(x_n,y_n)}$, with $a_n \geq 0$, $(x_n, y_n) \in \widetilde{M}$ and $\sum_{n=1}^{\infty} a_n = \|\mu\| = \|m\|$. As above we obtain $m = \sum_{n=1}^{\infty} a_n m_{x_n y_n}$. $\square$

Given a metric space (M, d) and a real number $\theta \in (0, 1)$, the **snowflake** M^θ is the metric space (M, d^θ). It is easy to check that d^θ is a metric on M.

Proposition 3.3 *Let $M = [0, 1]$ and $\theta \in (0, 1)$. Then there is a convex integral of molecules in $\mathcal{F}(M^\theta)$ that is not a convex series of molecules.*

3.2 Majorisable elements and uniformly discrete spaces

Definition 3.4 Let $m \in \mathcal{F}(M)$.

1. We say that m is **positive** if $\langle m, f\rangle \geq 0$ whenever $f \in \mathrm{Lip}_0(M)$ satisfies $f \geq 0$.
2. We say that $m \in \mathcal{F}(M)$ is **majorisable** if $m = m_1 - m_2$, where m_1, m_2 are positive.

Theorem 3.5 *Let $m \in \mathcal{F}(M)$. Then m is majorisable if and only if it is a convex integral of molecules with a representation $\mu \in \mathcal{M}_{\mathrm{op}}(\widetilde{M})$ satisfying*

$$\int_{\widetilde{M}} \frac{d(x, 0)}{d(x, y)} \, \mathrm{d}\mu(x, y) < \infty.$$

Definition 3.6 A metric space M is called **uniformly discrete** if there exists $r > 0$ such that $d(x, y) \geq r$ whenever $x, y \in M$, $x \neq y$.

Observe that all uniformly discrete metric spaces are scattered, and no infinite uniformly discrete space can be proper.

Corollary 3.7 *If M is uniformly discrete and bounded then every element of $\mathcal{F}(M)$ is a convex series of molecules.*

Proof. By [2, Theorem 6.2], every element of $\mathcal{F}(M)$ is majorisable. The conclusion follows from Theorem 3.5 and Proposition 3.2. $\square$

3.3 Not all free space elements are convex integrals of molecules

Theorem 3.8 below shows that it is common for free spaces to contain elements that fail to be convex integrals of molecules in a dramatic way.

Theorem 3.8 *Let M contain an isometric copy of a subset of $\mathbb{R}$ with positive Lebesgue measure. Then there exists $m \in \mathcal{F}(M)$ such that* $\operatorname{supp}(\mu) \cap \widetilde{M} = \varnothing$ *whenever $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ represents m.*

The following example is a special case of the above which contains the essential idea.

Example 3.9 Let $M = [0,1]$ and let $C \subseteq M$ be a nowhere dense 'fat Cantor set'. Then

$$[0,1] \setminus C = \bigcup_{n=1}^{\infty}(a_n, b_n),$$

where $(a_n, b_n) \subseteq [0,1]$ are pairwise disjoint open intervals such that $\sum_{n=1}^{\infty}(b_n - a_n) < 1$.
Set

$$m = m_{10} - \sum_{n=1}^{\infty}(b_n - a_n)m_{b_n a_n} \in \mathcal{F}([0,1]).$$

Then $\operatorname{supp}(\mu) \cap \widetilde{M} = \varnothing$ whenever $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ represents m.

To prove Example 3.9 we need the following lemma, which is an interesting result in its own right; among other things it is the first step in revealing the intimate connection between optimal de Leeuw representations and the idea of cyclical monotonicity from optimal transport theory; alas, there is not time in these lectures to explore this connection.

Lemma 3.10 *Let $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ represent $\psi \in \mathrm{Lip}_0(M)^*$ and suppose $\langle \psi, f \rangle = \|\psi\|$ for some $f \in S_{\mathrm{Lip}_0(M)}$. Then $\Phi f(\zeta) = 1$ whenever $\zeta \in \operatorname{supp}(\mu)$.*

Proof. Suppose $\Phi f(\zeta) < 1$ for some $\zeta \in \operatorname{supp}(\mu)$. By continuity, there exists $a < 1$ and open $U \ni \zeta$ such that $\Phi f \leq a$ on U. But then $\mu(U) > 0$ and

$$\|\mu\| = \|\psi\| = \langle \psi, f \rangle = \int_{\beta\widetilde{M}} \Phi f \,\mathrm{d}\mu \leq \int_{\beta\widetilde{M}\setminus U} \mathrm{d}\mu + a\int_U \mathrm{d}\mu < \|\mu\|. \qquad \square$$

Proof of Example 3.9. Let ν denote Lebesgue measure on $[0,1]$. First we show that $\|m\| = \nu(C) > 0$. Given $f \in B_{\mathrm{Lip}_0(M)}$, by absolute continuity of f we have $\|f'\|_\infty \leq 1$ and (recalling that $f(0) = 0$)

$$\begin{aligned}\langle m, f \rangle &= f(1) - \sum_{n=1}^{\infty}(f(b_n) - f(a_n)) \\ &= \int_0^1 f'(x)\,\mathrm{d}x - \sum_{n=1}^{\infty}\int_{a_n}^{b_n} f'(x)\,\mathrm{d}\nu(x) = \int_C f'(x)\,\mathrm{d}\nu(x) \leq \nu(C).\end{aligned}$$

Consequently $\|m\| \leq \nu(C)$.

Now define $g \in B_{\mathrm{Lip}_0(M)}$ by $g(x) = \int_0^x \mathbf{1}_C \, \mathrm{d}\nu = \nu([0,x] \cap C)$. Then $g' = \mathbf{1}_C$ ν-a.e. and

$$\langle m, g \rangle = \int_C g'(x) \, \mathrm{d}\nu(x) = \nu(C),$$

so $\|m\| = \langle m, g \rangle = \nu(C)$.

We claim that $(\Phi g)(x,y) < 1$ whenever $(x,y) \in \widetilde{M}$, giving $\mathrm{supp}(\mu) \cap \widetilde{M} = \varnothing$ by Lemma 3.10. Indeed, for $x > y$

$$(\Phi g)(x,y) = \frac{\nu([0,x] \cap C) - \nu([0,y] \cap C)}{x-y} = \frac{\nu([y,x] \cap C)}{x-y} < 1,$$

because C is nowhere dense, and $(\Phi g)(x,y) \le 0$ for $x < y$ as g is an increasing function. □

4 The extreme point conjecture for free spaces

4.1 New results

Theorem 4.1 *Let $m \in \mathrm{ext}\, B_{\mathcal{F}(M)}$, and let $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ represent m and satisfy $\mu(\widetilde{M}) > 0$. Then $m \in E$. In particular, this holds if m is a convex integral of molecules.*

Lemma 4.2 (Aliaga 22 [1, Lemma 10]) *Let $m \in \mathrm{ext}\, B_{\mathcal{F}(M)}$ and let $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ represent m. If $\lambda \in \mathcal{M}(\beta\widetilde{M})$ is such that $0 \le \lambda \le \mu$ and $\Phi^*\lambda \in \mathcal{F}(M)$, then $\Phi^*\lambda = \|\lambda\| \cdot m$.*

Proof. Note that $\lambda, \mu - \lambda \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ by Proposition 2.11(c). If either of them is 0 then we're done. Otherwise

$$m = \Phi^*\lambda + \Phi^*(\mu - \lambda) = \|\lambda\| \cdot \Phi^*\left(\frac{\lambda}{\|\lambda\|}\right) + \|\mu - \lambda\| \cdot \Phi^*\left(\frac{\mu - \lambda}{\|\mu - \lambda\|}\right)$$

is a convex combination of elements of $B_{\mathcal{F}(M)}$ by Proposition 2.11(b), hence $m = \Phi^*(\lambda / \|\lambda\|)$ by extremality. □

Proof of Theorem 4.1. By inner regularity of μ we have $\mu(K) > 0$ for some compact set $K \subseteq \widetilde{M}$. By a standard compactness argument, there exists $(x,y) \in K$ such that $\mu(U \times V) > 0$ whenever $U \ni x$ and $V \ni y$ are disjoint and open in M. We claim that $m = m_{xy}$, i.e. $\langle m, f \rangle = \Phi f(x,y)$ whenever $f \in \mathrm{Lip}_0(M)$. Given $f \in \mathrm{Lip}_0(M)$ and $\varepsilon > 0$, by continuity of Φf, there exist disjoint sets $U \ni x$, $V \ni y$, open in $\widetilde{M}$, such that

$$|\Phi f(x',y') - \Phi f(x,y)| < \varepsilon \quad \text{whenever } (x',y') \in U \times V.$$

Set $\lambda = \mu \restriction_{U \times V}$. Then $\lambda \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$, by Proposition 2.11 (d), and is concentrated on $U \times V \subseteq \widetilde{M}$, so $\Phi^*\lambda \in \mathcal{F}(M)$ by Proposition 2.13. Moreover, $\|\lambda\| = \mu(U \times V) > 0$ by the choice of (x,y). By Lemma 4.2 we conclude $m = \Phi^*\lambda / \|\lambda\|$. Thus

$$\begin{aligned}
|\langle m, f \rangle - \Phi f(x,y)| &= \left| \frac{1}{\|\lambda\|} \langle \Phi^*\lambda, f \rangle - \Phi f(x,y) \right| \\
&= \left| \frac{1}{\|\lambda\|} \int_{U \times V} \Phi f(x',y') \, \mathrm{d}\lambda(x',y') - \Phi f(x,y) \right| \\
&\le \frac{1}{\|\lambda\|} \int_{U \times V} |\Phi f(x',y') - \Phi f(x,y)| \, \mathrm{d}\lambda(x',y') < \frac{\mu(U \times V)\varepsilon}{\|\lambda\|} = \varepsilon.
\end{aligned}$$

It follows that $m = m_{xy}$ as claimed. □

Combining Theorem 3.5 and Corollary 3.7 with Theorem 4.1, we obtain a new case of the extreme point conjecture.

Corollary 4.3 *If an extreme point of $B_{\mathcal{F}(M)}$ is majorisable, then it is an elementary molecule. E.g. if M is uniformly discrete and bounded then* $\operatorname{ext} B_{\mathcal{F}(M)} \subseteq E$.

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