Lipschitz-free spaces and representing measures

XXII Lluís Santaló School, Santander, 2023

Richard J. Smith

University College Dublin

17–19 July

Lipschitz-free Banach spaces

Definition 1.1

1. Let (M, d) be a complete metric space with base point 0. Define $\mathrm{Lip}_0(M)$ to be the Banach space of all Lipschitz functions $f : M \to \mathbb{R}$ that vanish at 0, with norm

$$\|f\| := \mathrm{Lip}(f) = \sup \left\{ \frac{f(x) - f(y)}{d(x,y)} \ : \ x, y \in M,\ x \neq y \right\}.$$

Lipschitz-free Banach spaces

Definition 1.1

1. Let (M,d) be a complete metric space with base point 0. Define $\mathrm{Lip}_0(M)$ to be the Banach space of all Lipschitz functions $f : M \to \mathbb{R}$ that vanish at 0, with norm

$$\|f\| := \mathrm{Lip}(f) = \sup\left\{\frac{f(x)-f(y)}{d(x,y)} : x,y \in M,\, x \neq y\right\}.$$

2. Define $\widetilde{M} = \{(x,y) \in M^2 : x \neq y\}$ and the set $E = \left\{m_{xy} : (x,y) \in \widetilde{M}\right\} \subseteq S_{\mathrm{Lip}_0(M)^*}$ of **elementary molecules** m_{xy}, where

$$\langle m_{xy}, f\rangle = \frac{f(x)-f(y)}{d(x,y)}, \qquad f \in \mathrm{Lip}_0(M).$$

Lipschitz-free Banach spaces

Definition 1.1

1. Let (M, d) be a complete metric space with base point 0. Define $\mathrm{Lip}_0(M)$ to be the Banach space of all Lipschitz functions $f : M \to \mathbb{R}$ that vanish at 0, with norm

$$\|f\| := \mathrm{Lip}(f) = \sup\left\{\frac{f(x) - f(y)}{d(x, y)} \;:\; x, y \in M, \, x \neq y\right\}.$$

2. Define $\widetilde{M} = \{(x, y) \in M^2 \;:\; x \neq y\}$ and the set $E = \left\{m_{xy} \;:\; (x, y) \in \widetilde{M}\right\} \subseteq S_{\mathrm{Lip}_0(M)^*}$ of **elementary molecules** m_{xy}, where

$$\langle m_{xy}, f\rangle = \frac{f(x) - f(y)}{d(x, y)}, \qquad f \in \mathrm{Lip}_0(M).$$

3. Define the **(Lipschitz-) free** Banach space

$$\mathcal{F}(M) = \overline{\mathrm{span}}^{\|\cdot\|}(E) \subseteq \mathrm{Lip}_0(M)^*.$$

Convex series of molecules

Proposition 1.2

We have $B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}^{\|\cdot\|}(E)$.

Convex series of molecules

Proposition 1.2

We have $B_{\mathcal{F}(M)} = \overline{\text{conv}}^{\|\cdot\|}(E)$.

Exercise 1.3

Let X be a normed space and let $H \subseteq X$ with $B_X = \overline{\text{conv}}^{\|\cdot\|}(H)$. Show that, given $x \in X$ and $\varepsilon > 0$, there exist $x_n \in H$ and $a_n \geq 0$, $n \in \mathbb{N}$ satisfying

$$x = \sum_{n=1}^{\infty} a_n x_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n \leq \|x\| + \varepsilon.$$

Convex series of molecules

Corollary 1.4

Given $m \in \mathcal{F}(M)$ and $\varepsilon > 0$, there exist $(x_n, y_n) \in \widetilde{M}$ and $a_n \geq 0$, $n \in \mathbb{N}$, such that

$$m = \sum_{n=1}^{\infty} a_n m_{x_n y_n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n \leq \|m\| + \varepsilon.$$

Convex series of molecules

Corollary 1.4

Given $m \in \mathcal{F}(M)$ and $\varepsilon > 0$, there exist $(x_n, y_n) \in \widetilde{M}$ and $a_n \geq 0$, $n \in \mathbb{N}$, such that

$$m = \sum_{n=1}^{\infty} a_n m_{x_n y_n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n \leq \|m\| + \varepsilon.$$

We distinguish those elements of $\mathcal{F}(M)$ for which ε above can be set to 0.

Definition 1.5 (Aliaga, Rueda Zoca 20)

We say that $m \in \mathcal{F}(M)$ is a **convex series of molecules** if there exist $(x_n, y_n) \in \widetilde{M}$ and $a_n \geq 0$, $n \in \mathbb{N}$, such that

$$m = \sum_{n=1}^{\infty} a_n m_{x_n, y_n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n = \|m\| .$$

The extreme point conjecture

Conjecture 1.6 (Weaver)

Every extreme point of $B_{\mathcal{F}(M)}$ is an elementary molecule: $\operatorname{ext} B_{\mathcal{F}(M)} \subseteq E$.

The extreme point conjecture

Conjecture 1.6 (Weaver)

Every extreme point of $B_{\mathcal{F}(M)}$ is an elementary molecule: $\operatorname{ext} B_{\mathcal{F}(M)} \subseteq E$.

Proposition 1.7 (Aliaga, Pernecká, Petitjean, Procházka 20)

If $m \in \operatorname{ext} B_{\mathcal{F}(M)}$ is a convex series of molecules then $m \in E$.

The extreme point conjecture

Conjecture 1.6 (Weaver)

Every extreme point of $B_{\mathcal{F}(M)}$ is an elementary molecule: $\operatorname{ext} B_{\mathcal{F}(M)} \subseteq E$.

Proposition 1.7 (Aliaga, Pernecká, Petitjean, Procházka 20)

If $m \in \operatorname{ext} B_{\mathcal{F}(M)}$ is a convex series of molecules then $m \in E$.

Definition 1.8

The metric space M is **proper** if all of its closed bounded subsets are compact.

The extreme point conjecture

Conjecture 1.6 (Weaver)

Every extreme point of $B_{\mathcal{F}(M)}$ is an elementary molecule: $\mathrm{ext}\, B_{\mathcal{F}(M)} \subseteq E$.

Proposition 1.7 (Aliaga, Pernecká, Petitjean, Procházka 20)

If $m \in \mathrm{ext}\, B_{\mathcal{F}(M)}$ is a convex series of molecules then $m \in E$.

Definition 1.8

The metric space M is **proper** if all of its closed bounded subsets are compact.

Theorem 1.9 (Aliaga 22)

If M is proper then $\mathrm{ext}\, B_{\mathcal{F}(M)} \subseteq E$.

Optimal de Leeuw representations

Definition 2.1

Given a Tychonoff (completely regular Hausdorff) topological space X, the **Stone-Čech compactification** βX of X is a Hausdorff compactification of X characterised (up to homeomorphism) by the **extension property**: any continuous map $f : X \to K$ (K compact Hausdorff) can be extended uniquely to a continuous map $f : \beta X \to K$.

Optimal de Leeuw representations

Definition 2.1

Given a Tychonoff (completely regular Hausdorff) topological space X, the **Stone-Čech compactification** βX of X is a Hausdorff compactification of X characterised (up to homeomorphism) by the **extension property**: any continuous map $f : X \to K$ (K compact Hausdorff) can be extended uniquely to a continuous map $f : \beta X \to K$.

Definition 2.2

Define $\Phi : \mathrm{Lip}_0(M) \to C(\beta\widetilde{M})$ by first setting

$$(\Phi f)(x,y) = \frac{f(x) - f(y)}{d(x,y)} = \langle m_{xy}, f \rangle \,, \qquad (x,y) \in \widetilde{M},$$

and then extending continuously to its Stone-Čech compactification $\beta\widetilde{M}$.

Optimal de Leeuw representations

Definition 2.1

Given a Tychonoff (completely regular Hausdorff) topological space X, the **Stone-Čech compactification** βX of X is a Hausdorff compactification of X characterised (up to homeomorphism) by the **extension property**: any continuous map $f : X \to K$ (K compact Hausdorff) can be extended uniquely to a continuous map $f : \beta X \to K$.

Definition 2.2

Define $\Phi : \mathrm{Lip}_0(M) \to C(\beta\widetilde{M})$ by first setting

$$(\Phi f)(x,y) = \frac{f(x) - f(y)}{d(x,y)} = \langle m_{xy}, f\rangle, \qquad (x,y) \in \widetilde{M},$$

and then extending continuously to its Stone-Čech compactification $\beta\widetilde{M}$.

We call both Φ and its dual $\Phi^* : C(\beta\widetilde{M})^* \equiv \mathcal{M}(\beta\widetilde{M}) \to \mathrm{Lip}_0(M)^*$ **de Leeuw transforms**.

Optimal de Leeuw representations

Exercise 2.3

Show that Φ is an isometric embedding and Φ^* is a quotient map.

Optimal de Leeuw representations

Exercise 2.3

Show that Φ is an isometric embedding and Φ^* is a quotient map.

Definition 2.4

Given $\psi \in \mathrm{Lip}_0(M)^*$, we call $\mu \in \mathcal{M}(\beta\widetilde{M})$ a **(de Leeuw) representation** of ψ if $\Phi^*\mu = \psi$.

Optimal de Leeuw representations

Exercise 2.3

Show that Φ is an isometric embedding and Φ^* is a quotient map.

Definition 2.4

Given $\psi \in \mathrm{Lip}_0(M)^*$, we call $\mu \in \mathcal{M}(\beta\widetilde{M})$ a **(de Leeuw) representation** of ψ if $\Phi^*\mu = \psi$.

We have $\|\psi\| \leq \|\mu\|$ always; we focus on those **positive** μ for which $\|\Phi^*\mu\| = \|\mu\|$.

Optimal de Leeuw representations

Exercise 2.3

Show that Φ is an isometric embedding and Φ^* is a quotient map.

Definition 2.4

Given $\psi \in \mathrm{Lip}_0(M)^*$, we call $\mu \in \mathcal{M}(\beta\widetilde{M})$ a **(de Leeuw) representation** of ψ if $\Phi^*\mu = \psi$.

We have $\|\psi\| \leq \|\mu\|$ always; we focus on those **positive** μ for which $\|\Phi^*\mu\| = \|\mu\|$.

Definition 2.5

Define the set of **optimal representations**

$$\mathcal{M}_{\mathrm{op}}(\beta\widetilde{M}) = \left\{\mu \in \mathcal{M}(\beta\widetilde{M}) \; : \; \mu \geq 0 \text{ and } \|\Phi^*\mu\| = \|\mu\|\right\}.$$

Examples of optimal representations

Example 2.6

Let $(x, y) \in \widetilde{M}$. Then $\Phi^* \delta_{(x,y)} = m_{xy}$ and $\delta_{(x,y)} \in \mathcal{M}_{\mathrm{op}}(\beta \widetilde{M})$ as $\left\| \delta_{(x,y)} \right\| = 1 = \| m_{xy} \|$.

Examples of optimal representations

Example 2.6

Let $(x,y) \in \widetilde{M}$. Then $\Phi^*\delta_{(x,y)} = m_{xy}$ and $\delta_{(x,y)} \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ as $\left\|\delta_{(x,y)}\right\| = 1 = \|m_{xy}\|$.

Example 2.7

Let $M := [0,1]$ have base point 0. Define positive $\mu_n \in \mathcal{M}(\beta\widetilde{M})^+$, $n \geq 0$, by

$$\mu_n = \frac{1}{2^n}\sum_{i=1}^{2^n} \delta_{\left(\frac{i}{2^n}, \frac{i-1}{2^n}\right)}.$$

Examples of optimal representations

Example 2.6

Let $(x,y) \in \widetilde{M}$. Then $\Phi^*\delta_{(x,y)} = m_{xy}$ and $\delta_{(x,y)} \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ as $\left\|\delta_{(x,y)}\right\| = 1 = \|m_{xy}\|$.

Example 2.7

Let $M := [0,1]$ have base point 0. Define positive $\mu_n \in \mathcal{M}(\beta\widetilde{M})^+$, $n \geq 0$, by

$$\mu_n = \frac{1}{2^n}\sum_{i=1}^{2^n} \delta_{\left(\frac{i}{2^n},\frac{i-1}{2^n}\right)}.$$

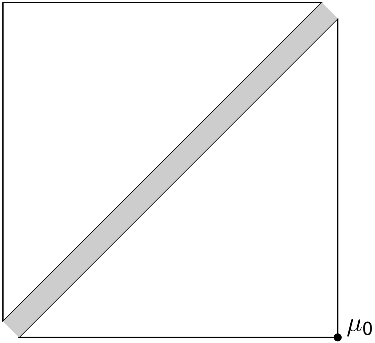
Then $\mu_n \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$, with $\Phi^*\mu_n = m_{10}$: $\|\mu_n\| = 1 = \|m_{10}\|$ and

$$\langle \Phi^*\mu_n, f\rangle = \langle \mu_n, \Phi f\rangle = \frac{1}{2^n}\sum_{i=1}^{2^n} \frac{f\left(\frac{i}{2^n}\right) - f\left(\frac{i-1}{2^n}\right)}{2^{-n}} = f(1) - f(0) = \langle m_{10}, f\rangle, \qquad f \in \mathrm{Lip}_0(M),$$

giving $\Phi^*\mu_n = m_{10}$.

Examples of optimal representations

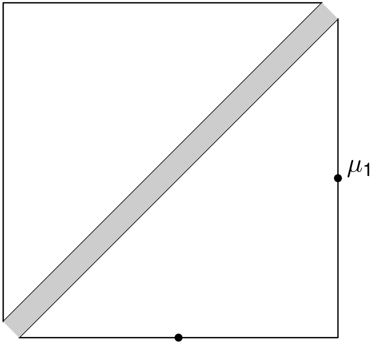
Each $\mu_n \in \mathcal{M}(\beta\widetilde{M})^+$, $n \geq 0$, is an optimal representation of m_{10}:



The above is a depiction of $\beta\widetilde{M}$, with the shaded area representing the 'diagonal' $\beta\widetilde{M} \setminus \widetilde{M}$.

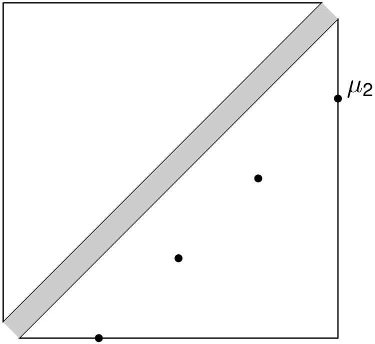
Examples of optimal representations

Each $\mu_n \in \mathcal{M}(\beta\widetilde{M})^+$, $n \geq 0$, is an optimal representation of m_{10}:



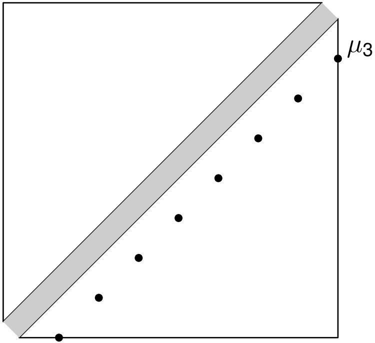
Examples of optimal representations

Each $\mu_n \in \mathcal{M}(\beta\widetilde{M})^+$, $n \geq 0$, is an optimal representation of m_{10}:



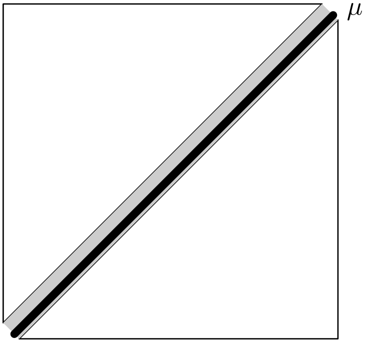
Examples of optimal representations

Each $\mu_n \in \mathcal{M}(\beta\widetilde{M})^+$, $n \geq 0$, is an optimal representation of m_{10}:



Examples of optimal representations

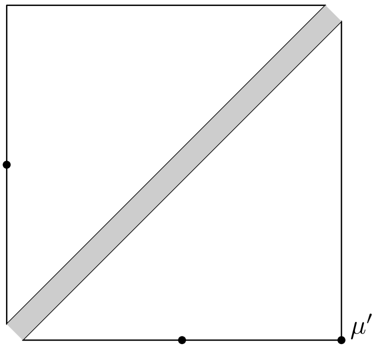
Each $\mu_n \in \mathcal{M}(\beta\widetilde{M})^+$, $n \geq 0$, is an optimal representation of m_{10}:



Any w^*-accumulation point μ of (μ_n) is also an optimal representation of m_{10}; any such point is supported entirely on the 'diagonal' $\beta\widetilde{M} \setminus \widetilde{M}$.

Examples of optimal representations

Each $\mu_n \in \mathcal{M}(\beta\widetilde{M})^+$, $n \geq 0$, is an optimal representation of m_{10}:



On the other hand, the representation $\mu' := \delta_{(1,0)} + \delta_{(\frac{1}{2},0)} + \delta_{(0,\frac{1}{2})}$ is not optimal: $\|\mu'\| = 3$, with the mass reflected in the diagonal 'cancelling out' because $\Phi f(x,y) = -\Phi f(y,x)$, $(x,y) \in \widetilde{M}$.

Basic properties of optimal representations

Definition 2.8

Define the **reflection map** $\mathfrak{r} : \beta\widetilde{M} \to \beta\widetilde{M}$ by setting $r(x, y) = (y, x)$, $(x, y) \in \widetilde{M}$, and then extending continuously to $\beta\widetilde{M}$.

Basic properties of optimal representations

Definition 2.8

Define the **reflection map** $\mathfrak{r} : \beta\widetilde{M} \to \beta\widetilde{M}$ by setting $r(x,y) = (y,x)$, $(x,y) \in \widetilde{M}$, and then extending continuously to $\beta\widetilde{M}$. Then define the isometry $R : C(\beta\widetilde{M}) \to C(\beta\widetilde{M})$ by $Rf = f \circ \mathfrak{r}$.

Basic properties of optimal representations

Definition 2.8

Define the **reflection map** $\mathfrak{r} : \beta\widetilde{M} \to \beta\widetilde{M}$ by setting $r(x, y) = (y, x)$, $(x, y) \in \widetilde{M}$, and then extending continuously to $\beta\widetilde{M}$. Then define the isometry $R : C(\beta\widetilde{M}) \to C(\beta\widetilde{M})$ by $Rf = f \circ \mathfrak{r}$.

Exercise 2.9

Show that $R\Phi = -\Phi$, and thus $\Phi^* R^* = -\Phi^*$.

Basic properties of optimal representations

Definition 2.10

Let $\mu \in \mathcal{M}(\beta\widetilde{M})$ and let $E \subseteq \beta\widetilde{M}$ be Borel. We say that μ is **concentrated** on E if $\mu(A) = \mu(A \cap E)$ whenever $A \subseteq \beta\widetilde{M}$ is Borel.

Basic properties of optimal representations

Definition 2.10

Let $\mu \in \mathcal{M}(\beta\widetilde{M})$ and let $E \subseteq \beta\widetilde{M}$ be Borel. We say that μ is **concentrated** on E if $\mu(A) = \mu(A \cap E)$ whenever $A \subseteq \beta\widetilde{M}$ is Borel. We define the **restriction** $\mu\restriction_E$ of μ to E by $\mu\restriction_E(A) = \mu(A \cap E)$, A Borel.

Basic properties of optimal representations

Definition 2.10

Let $\mu \in \mathcal{M}(\beta\widetilde{M})$ and let $E \subseteq \beta\widetilde{M}$ be Borel. We say that μ is **concentrated** on E if $\mu(A) = \mu(A \cap E)$ whenever $A \subseteq \beta\widetilde{M}$ is Borel. We define the **restriction** $\mu\restriction_E$ of μ to E by $\mu\restriction_E(A) = \mu(A \cap E)$, A Borel.

Proposition 2.11

1. For any $\psi \in \mathrm{Lip}_0(M)^*$ there is $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ such that $\Phi^*\mu = \psi$.

Basic properties of optimal representations

Definition 2.10

Let $\mu \in \mathcal{M}(\beta\widetilde{M})$ and let $E \subseteq \beta\widetilde{M}$ be Borel. We say that μ is **concentrated** on E if $\mu(A) = \mu(A \cap E)$ whenever $A \subseteq \beta\widetilde{M}$ is Borel. We define the **restriction** $\mu\restriction_E$ of μ to E by $\mu\restriction_E(A) = \mu(A \cap E)$, A Borel.

Proposition 2.11

1. For any $\psi \in \mathrm{Lip}_0(M)^*$ there is $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ such that $\Phi^*\mu = \psi$.
2. If $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ then $c \cdot \mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ for every $c \geq 0$.

Basic properties of optimal representations

Definition 2.10

Let $\mu \in \mathcal{M}(\beta\widetilde{M})$ and let $E \subseteq \beta\widetilde{M}$ be Borel. We say that μ is **concentrated** on E if $\mu(A) = \mu(A \cap E)$ whenever $A \subseteq \beta\widetilde{M}$ is Borel. We define the **restriction** $\mu\restriction_E$ of μ to E by $\mu\restriction_E(A) = \mu(A \cap E)$, A Borel.

Proposition 2.11

1. For any $\psi \in \mathrm{Lip}_0(M)^*$ there is $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ such that $\Phi^*\mu = \psi$.
2. If $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ then $c \cdot \mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ for every $c \geq 0$.
3. If $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ and $\lambda \in \mathcal{M}(\beta\widetilde{M})$ satisfies $0 \leq \lambda \leq \mu$, then $\lambda \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$.

Basic properties of optimal representations

Definition 2.10

Let $\mu \in \mathcal{M}(\beta\widetilde{M})$ and let $E \subseteq \beta\widetilde{M}$ be Borel. We say that μ is **concentrated** on E if $\mu(A) = \mu(A \cap E)$ whenever $A \subseteq \beta\widetilde{M}$ is Borel. We define the **restriction** $\mu\restriction_E$ of μ to E by $\mu\restriction_E(A) = \mu(A \cap E)$, A Borel.

Proposition 2.11

1. For any $\psi \in \mathrm{Lip}_0(M)^*$ there is $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ such that $\Phi^*\mu = \psi$.
2. If $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ then $c \cdot \mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ for every $c \geq 0$.
3. If $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ and $\lambda \in \mathcal{M}(\beta\widetilde{M})$ satisfies $0 \leq \lambda \leq \mu$, then $\lambda \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$.
4. If $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ and $E \subseteq \beta\widetilde{M}$ is Borel then $\mu\restriction_E \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$.

The support of a measure

Definition 2.12

Let $\mu \in \mathcal{M}(\beta\widetilde{M})$ be positive. Define

$$\text{supp}(\mu) = \left\{\zeta \in \beta\widetilde{M} \;:\; \mu(U) > 0 \text{ whenever } U \ni \zeta \text{ is open in } \beta\widetilde{M}\right\}$$

$$\text{and } \text{supp}_{\widetilde{M}}(\mu) = \left\{(x,y) \in \widetilde{M} \;:\; \mu(U) > 0 \text{ whenever } U \ni (x,y) \text{ is open in } \widetilde{M}\right\}.$$

The support of a measure

Definition 2.12

Let $\mu \in \mathcal{M}(\beta\widetilde{M})$ be positive. Define

$$\text{supp}(\mu) = \left\{\zeta \in \beta\widetilde{M} \;:\; \mu(U) > 0 \text{ whenever } U \ni \zeta \text{ is open in } \beta\widetilde{M}\right\}$$

$$\text{and } \text{supp}_{\widetilde{M}}(\mu) = \left\{(x,y) \in \widetilde{M} \;:\; \mu(U) > 0 \text{ whenever } U \ni (x,y) \text{ is open in } \widetilde{M}\right\}.$$

For such μ it holds that μ is concentrated on $\text{supp}(\mu)$: $\|\mu\| = \mu(\text{supp}(\mu))$, and if μ is concentrated on $\widetilde{M}$ then $\text{supp}_{\widetilde{M}}(\mu) = \text{supp}(\mu) \cap \widetilde{M}$.

Convex integrals of molecules

Proposition 2.13

If $\mu \in \mathcal{M}(\beta\widetilde{M})$ then

$$\Phi^*(\mu\restriction_{\widetilde{M}}) = \int_{\widetilde{M}} m_{xy}\, \mathrm{d}\mu(x,y)$$

as a Bochner integral on $\mathcal{F}(M)$.

Convex integrals of molecules

Proposition 2.13

If $\mu \in \mathcal{M}(\beta\widetilde{M})$ then

$$\Phi^*(\mu\restriction_{\widetilde{M}}) = \int_{\widetilde{M}} m_{xy} \,\mathrm{d}\mu(x,y)$$

as a Bochner integral on $\mathcal{F}(M)$.

Definition 2.14

We say that $m \in \mathcal{F}(M)$ is a **convex integral of molecules** if $m = \Phi^*\mu$ for some $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ concentrated on $\widetilde{M}$.

Relationship with convex series of molecules

Proposition 3.1

Every convex series of molecules is a convex integral of molecules.

Relationship with convex series of molecules

Proposition 3.1

Every convex series of molecules is a convex integral of molecules.

Proposition 3.2

If M is scattered, then every convex integral of molecules in $\mathcal{F}(M)$ is also a convex series of molecules.

Relationship with convex series of molecules

Proposition 3.1

Every convex series of molecules is a convex integral of molecules.

Proposition 3.2

If M is scattered, then every convex integral of molecules in $\mathcal{F}(M)$ is also a convex series of molecules.

Proposition 3.3

Let $M = [0, 1]$ and $\theta \in (0, 1)$. Then there is a convex integral of molecules in $\mathcal{F}(M^\theta)$ that is not a convex series of molecules.

Majorisable elements

Definition 3.4

Let $m \in \mathcal{F}(M)$.

1. We say that m is **positive** if $\langle m, f \rangle \geq 0$ whenever $f \in \mathrm{Lip}_0(M)$ satisfies $f \geq 0$.

Majorisable elements

Definition 3.4

Let $m \in \mathcal{F}(M)$.

1. We say that m is **positive** if $\langle m, f \rangle \geq 0$ whenever $f \in \mathrm{Lip}_0(M)$ satisfies $f \geq 0$.
2. We say that $m \in \mathcal{F}(M)$ is **majorisable** if $m = m_1 - m_2$, where m_1, m_2 are positive.

Majorisable elements

Definition 3.4

Let $m \in \mathcal{F}(M)$.

1. We say that m is **positive** if $\langle m, f \rangle \geq 0$ whenever $f \in \mathrm{Lip}_0(M)$ satisfies $f \geq 0$.
2. We say that $m \in \mathcal{F}(M)$ is **majorisable** if $m = m_1 - m_2$, where m_1, m_2 are positive.

Theorem 3.5

Let $m \in \mathcal{F}(M)$. Then m is majorisable if and only if it is a convex integral of molecules with a representation $\mu \in \mathcal{M}_{\mathrm{op}}(\widetilde{M})$ satisfying

$$\int_{\widetilde{M}} \frac{d(x,0)}{d(x,y)} \, \mathrm{d}\mu(x,y) < \infty.$$

Uniformly discrete spaces

Definition 3.6

A metric space M is called **uniformly discrete** if there exists $r > 0$ such that $d(x, y) \geq r$ whenever $x, y \in M$, $x \neq y$.

Uniformly discrete spaces

Definition 3.6

A metric space M is called **uniformly discrete** if there exists $r > 0$ such that $d(x, y) \geq r$ whenever $x, y \in M$, $x \neq y$.

Corollary 3.7

If M is uniformly discrete and bounded then every element of $\mathcal{F}(M)$ is a convex series of molecules.

Not all free space elements are convex integrals of molecules

Theorem 3.8

Let M contain an isometric copy of a subset of $\mathbb{R}$ with positive Lebesgue measure. Then there exists $m \in \mathcal{F}(M)$ such that $\mathrm{supp}(\mu) \cap \widetilde{M} = \varnothing$ whenever $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ represents m.

Fat Cantor sets

Example 3.9

Let $M = [0,1]$ and let $C \subseteq M$ be a nowhere dense 'fat Cantor set'. Then

$$[0,1] \setminus C = \bigcup_{n=1}^{\infty} (a_n, b_n),$$

where $(a_n, b_n) \subseteq [0,1]$ are pairwise disjoint open intervals such that $\sum_{n=1}^{\infty} (b_n - a_n) < 1$.

Fat Cantor sets

Example 3.9

Let $M = [0,1]$ and let $C \subseteq M$ be a nowhere dense 'fat Cantor set'. Then

$$[0,1] \setminus C = \bigcup_{n=1}^{\infty} (a_n, b_n),$$

where $(a_n, b_n) \subseteq [0,1]$ are pairwise disjoint open intervals such that $\sum_{n=1}^{\infty}(b_n - a_n) < 1$. Set

$$m = m_{10} - \sum_{n=1}^{\infty} (b_n - a_n) m_{b_n a_n} \in \mathcal{F}([0,1]).$$

Fat Cantor sets

Example 3.9

Let $M = [0,1]$ and let $C \subseteq M$ be a nowhere dense 'fat Cantor set'. Then

$$[0,1] \setminus C = \bigcup_{n=1}^{\infty} (a_n, b_n),$$

where $(a_n, b_n) \subseteq [0,1]$ are pairwise disjoint open intervals such that $\sum_{n=1}^{\infty}(b_n - a_n) < 1$. Set

$$m = m_{10} - \sum_{n=1}^{\infty}(b_n - a_n) m_{b_n a_n} \in \mathcal{F}([0,1]).$$

Then $\mathrm{supp}(\mu) \cap \widetilde{M} = \varnothing$ whenever $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ represents m.

Fat Cantor sets

Example 3.9

Let $M = [0,1]$ and let $C \subseteq M$ be a nowhere dense 'fat Cantor set'. Then

$$[0,1] \setminus C = \bigcup_{n=1}^{\infty} (a_n, b_n),$$

where $(a_n, b_n) \subseteq [0,1]$ are pairwise disjoint open intervals such that $\sum_{n=1}^{\infty}(b_n - a_n) < 1$. Set

$$m = m_{10} - \sum_{n=1}^{\infty} (b_n - a_n) m_{b_n a_n} \in \mathcal{F}([0,1]).$$

Then $\mathrm{supp}(\mu) \cap \widetilde{M} = \varnothing$ whenever $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ represents m.

Lemma 3.10

Let $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ represent $\psi \in \mathrm{Lip}_0(M)^*$ and suppose $\langle \psi, f \rangle = \|\psi\|$ for some $f \in S_{\mathrm{Lip}_0(M)}$. Then $\Phi f(\zeta) = 1$ whenever $\zeta \in \mathrm{supp}(\mu)$.

The extreme point conjecture for free spaces

Theorem 4.1

Let $m \in \operatorname{ext} B_{\mathcal{F}(M)}$, and let $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ represent m and satisfy $\mu(\widetilde{M}) > 0$. Then $m \in E$. In particular, this holds if m is a convex integral of molecules.

The extreme point conjecture for free spaces

Theorem 4.1

Let $m \in \operatorname{ext} B_{\mathcal{F}(M)}$, and let $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ represent m and satisfy $\mu(\widetilde{M}) > 0$. Then $m \in E$. In particular, this holds if m is a convex integral of molecules.

Lemma 4.2 (Aliaga 22)

Let $m \in \operatorname{ext} B_{\mathcal{F}(M)}$ and let $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ represent m. If $\lambda \in \mathcal{M}(\beta\widetilde{M})$ is such that $0 \leq \lambda \leq \mu$ and $\Phi^*\lambda \in \mathcal{F}(M)$, then $\Phi^*\lambda = \|\lambda\| \cdot m$.

The extreme point conjecture for free spaces

Theorem 4.1

Let $m \in \operatorname{ext} B_{\mathcal{F}(M)}$, and let $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ represent m and satisfy $\mu(\widetilde{M}) > 0$. Then $m \in E$. In particular, this holds if m is a convex integral of molecules.

Lemma 4.2 (Aliaga 22)

Let $m \in \operatorname{ext} B_{\mathcal{F}(M)}$ and let $\mu \in \mathcal{M}_{\mathrm{op}}(\beta\widetilde{M})$ represent m. If $\lambda \in \mathcal{M}(\beta\widetilde{M})$ is such that $0 \leq \lambda \leq \mu$ and $\Phi^*\lambda \in \mathcal{F}(M)$, then $\Phi^*\lambda = \|\lambda\| \cdot m$.

Corollary 4.3

If an extreme point of $B_{\mathcal{F}(M)}$ is majorisable, then it is an elementary molecule. E.g. if M is uniformly discrete and bounded then $\operatorname{ext} B_{\mathcal{F}(M)} \subseteq E$.