universität innsbruck

~~Morning~~ tales

Norming

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Linear and Nonlinear Analysis in Banach spaces

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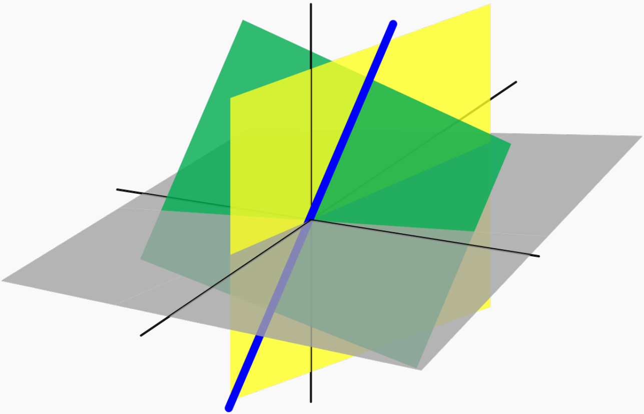
Joke fail

Yes, I did ask Sheldon to move my talk to the morning

Once upon a time in Linear Algebra



Some basis stuff

- Every Banach space $\mathcal{X}$ has a linear basis $\{v_\alpha\}_{\alpha\in\Gamma}$.
- Even when $\mathcal{X}$ is separable, the index set Γ is uncountable.
  - So, linear bases don't generalise complete orthonormal systems.
- Moreover, the linear functionals $\sum c_\alpha v_\alpha \mapsto c_\alpha$ are never continuous.
- A sequence $(e_n)_{n=1}^\infty$ is a **Schauder basis** if for all $x \in \mathcal{X}$ there are unique scalars $(x_n)_{n=1}^\infty$ with

$$x = \sum_{n=1}^\infty x_n e_n \qquad \text{(the series converges in } \mathcal{X}\text{).}$$

- The **coordinate functionals** $\varphi_n \colon \sum x_n e_n \mapsto x_n$ are continuous.
- Two drawbacks:
  - Schauder bases can only exist in separable spaces.
  - **Enflo ('73).** Not every separable Banach space has a Schauder basis.

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M for Markushevich

If $(e_n)_{n=1}^{\infty}$ is a Schauder basis and $(\varphi_n)_{n=1}^{\infty}$ are the coordinate functionals:

(i) $\langle \varphi_k, e_n \rangle = \delta_{k,n}$,

(ii) $\overline{\text{span}}\{e_n\} = \mathcal{X}$,

(iii) $\overline{\text{span}}^{w^*}\{\varphi_n\} = \mathcal{X}^*$.

- Drawback: $\sum_{n=1}^{\infty} \langle \varphi_n, x \rangle e_n$ might not converge!
- Advantages:
  - **Markushevich ('43).** Every separable Banach space has an M-basis.
  - The definition extends to all Banach spaces (just change label!).

Example: The trigonometric system $\{t \mapsto e^{ikt}\}_{k\in\mathbb{Z}}$ is not a Schauder basis of $\mathcal{C}(\mathbb{T})$ (or $L^1(\mathbb{T})$), but it is an M-basis.

- **Johnson ('70).** ℓ_∞ has no M-basis.

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A system $\{e_n; \varphi_n\}_{n=1}^{\infty} \subseteq \mathcal{X} \times \mathcal{X}^*$ is a **Markushevich basis** (**M-basis**) if:

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Let us welcome the main character

We actually have more:

- If $\mathcal{X}^*$ is separable, $\mathcal{X}$ admits an M-basis with $\overline{\operatorname{span}}\{\varphi_\alpha\} = \mathcal{X}^*$.
- Every separable Banach space, for every $\varepsilon > 0$, admits an M-basis $\{e_n; \varphi_n\}_{n=1}^{\infty}$ with $\|e_n\| \cdot \|\varphi_n\| \leqslant 1 + \varepsilon$.
- Every separable Banach space admits a 1-norming M-basis.

A subspace $\mathcal{Z}$ of $\mathcal{X}^*$ is **λ-norming** $(0 < \lambda \leqslant 1)$ if

$$\lambda\|x\| \leqslant \sup\{|\langle \varphi, x\rangle| \colon \varphi \in \mathcal{Z}, \|\varphi\| \leqslant 1\}.$$

Plainly, $\mathcal{X}^*$ is 1-norming, by the Hahn–Banach theorem.

Definition. An M-basis $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$ is **λ-norming** $(0 < \lambda \leqslant 1)$ if $\operatorname{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is a λ-norming subspace, namely if

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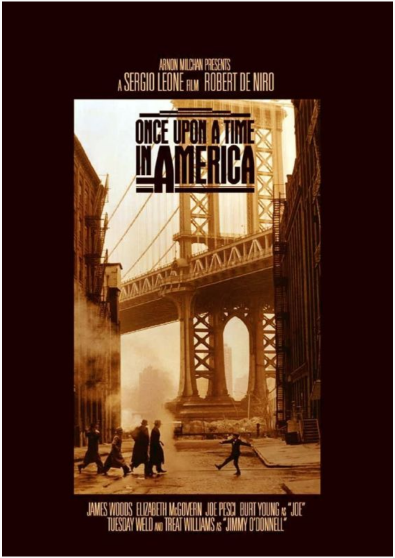

"Do WGC spaces!"
Viktor Klee → Vaclav Zizler, '69

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- More recent results: WCG spaces, or spaces with norming M-basis, are **Plichko**. And Plichko spaces have a PRI and a LUR norm.

Theorem (Hájek, Advances '19)

There exists a WCG $\mathcal{C}(\mathcal{K})$ space with no norming M-basis.

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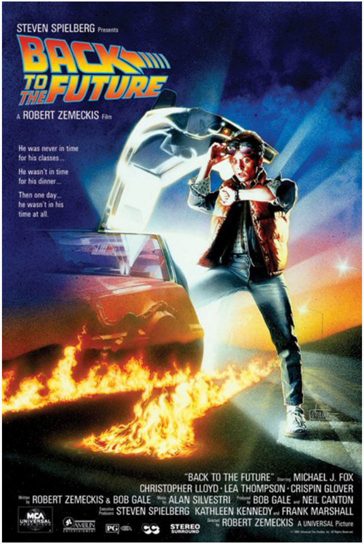
Back to the future



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(A few, recent) References

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T. Russo and J. Somaglia
Banach spaces of continuous functions without norming Markushevich bases
Mathematika (in press), `arXiv:2305.11737`.

So, in the end, norming or morning?

Thamk you for your attemtiom!