Geometry of tensor products and bilinear mappings in Banach spaces III

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Theorem

If $P : X \longrightarrow Z$ and $Q : Y \longrightarrow W$ are quotient operators, then so is $P \otimes Q : X\widehat{\otimes}_\pi Y \longrightarrow Z\widehat{\otimes}_\pi W$.

Description of elements of the $X\widehat{\otimes}_\pi Y$

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Let X be a normed space and call $\tilde{X}$ to its completion.

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We can assume $\tilde{x} \in B_{\tilde{X}}$. Let $m \in \mathbb{N}$. Take $x_1 \in B_X$ with $\|x - x_1\| < \frac{1}{2^m}$. Since $2^m(x - x_1) \in B_X$ find x_2 s.t. $\|2^m(x - x_1) - x_2\| < \frac{1}{2}$, so $\|x - x_1 - \frac{1}{2^m} x_2\| < \frac{1}{2^{m+1}}$. Inductively there exists (x_n) such that

$$\left\| \tilde{x} - x_1 - \sum_{k=1}^{n} \frac{1}{2^{m+k-1}} x_k \right\| < \frac{1}{2^{m+n-1}}.$$

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- P, Q quotient operators, for every $n \in \mathbb{N}$ there exists $x_n \in X$, $y_n \in Y$ with $P(x_n) = \frac{z_n}{\|z_n\|}$, $\|x_n\| < 1 + \varepsilon$ and $P(y_n) = \frac{w_n}{\|w_n\|}$, $\|y_n\| < 1 + \varepsilon$.

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- $(P \otimes Q)(\sum_{n=1}^\infty \lambda_n x_n \otimes y_n) = \sum_{n=1}^\infty \lambda_n \frac{z_n}{\|z_n\|} \otimes \frac{w_n}{\|w_n\|} = v$.

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Given $z \in X\widehat{\otimes}_\pi Y$, *then*

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We say $u \in X\widehat{\otimes}_\pi Y$ attains its projective norm if there exists a representation $u = \sum_{n=1}^{\infty} x_n \otimes y_n$ s.t. $\|u\|_\pi = \sum_{n=1}^{\infty}\|x_n\|\|y_n\|$.

$NA_\pi(X\widehat{\otimes}_\pi Y)$ stands for the elements attaining its projective norm.

Tensors which attain its projective norm

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Let X and Y be two Banach spaces. Let $z \in X\widehat{\otimes}_\pi Y$ such that

$$z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n$$

for suitable $(\lambda_n) \in \mathbb{R}^+$, $x_n \in S_X$ and $y_n \in S_Y$.

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1. $\|z\| = \sum_{n=1}^{\infty} \lambda_n$.
2. *For every $B \in \mathcal{B}(X \times Y)$ such that $B(z) = \|z\|$ it follows $B(x_n, y_n) = 1$ holds for every $n \in \mathbb{N}$.*

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A soft convexity argument.

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- If every element of $X\widehat{\otimes}_\pi Y$ attains its projective norm, then any bilinear form $B \in (X\widehat{\otimes}_\pi Y)^*$ with $\|B\| = 1$ and which attains its norm as functional acting on $(X\widehat{\otimes}_\pi Y)$ satisfies that $B(x, y) = 1$ holds for some $x \in S_X$ and $y \in S_Y$ (B attains its norm as bilinear map).

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This will be central for our main density result.

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Let X be a Banach space. We will say that X has the *metric π-property* if given $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subseteq S_X$ a finite collection in the sphere, then we can find a finite dimensional 1-complemented subspace $M \subseteq X$ such that for each $i \in \{1, \ldots, n\}$ there exists $x_i' \in M$ with $\|x_i - x_i'\| < \varepsilon$.

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3. Absolute sums of spaces with metric-π has the metric-π.
4. The projective tensor product of spaces with the metric-π has the metric-π.

Metric π-property and the density tensor attaining its norm

Theorem

If X and Y have the metric π-property then $NA_\pi(X\widehat{\otimes}_\pi Y)$ is dense in $X\widehat{\otimes}_\pi Y$.

Sketch:

- By density, take $u = \sum_{i=1}^n x_i \otimes y_i \in X\widehat{\otimes}_\pi Y$ arbitrary.

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- By the metric π we can find 1-complemented subspaces $E \subseteq X$ and $F \subseteq Y$ and, for every i, we can find $x_i' \in E$ and $y_i' \in F$ such that $x_i' \approx x_i, y_i' \approx y_i$.

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- $u'\in E\widehat{\otimes}_\pi F$, so it attains its projective norm.

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- $u'\in E\widehat{\otimes}_\pi F$, so it attains its projective norm. So we can write $u'=\sum_{i=1}^m a_i\otimes b_i$ and $\|u'\|_{E\widehat{\otimes}_\pi F}=\sum_{i=1}^m\|a_i\|\|b_i\|$.
- $\|u'\|_{X\widehat{\otimes}_\pi Y}=\|u'\|_{E\widehat{\otimes}_\pi F}=\sum_{i=1}^m\|a_i\|\|b_i\|$ since $E\widehat{\otimes}_\pi F\subseteq X\widehat{\otimes}_\pi Y$ isometrically (and even 1-complemented).

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Non-norm attaining tensors may be dense!

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Question 3

May $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ be residual or even contain an open dense set?

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Happy birthday!