Geometry of tensor products and bilinear mappings in Banach spaces II

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Junta de Andalucía

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What about the converse?

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Theorem

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1. $\|u\|_{W\widehat{\otimes}_\pi Z} \leq C\|u\|_{X\widehat{\otimes}_\pi Y}$ *for every* $u \in W \otimes Z$.

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(1)⇒(2). Set $B \in B(W \times Z)$. Given $u \in W \otimes Z$

$$B(u) \leq \|B\|\|u\|_{W\widehat{\otimes}_\pi Z} \leq \|B\|C\|u\|_{X\widehat{\otimes}_\pi Y}.$$

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Then B acts linear and continuously on $(W \otimes Z, \|\cdot\|_{X\widehat{\otimes}_\pi Y})$ and its norm is $\leq C\|B\|$. By Hahn-Banach theorem there exists $\tilde{B} \in (X\widehat{\otimes}_\pi Y)^* = B(X \times Y)$ extending B and norm preserving.

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$$\|u\|_{W\widehat{\otimes}_\pi Z} = \tilde{B}(u) \leq C\|u\|_{X\widehat{\otimes}_\pi Y}.$$

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Let Y Banach space and $T : W \longrightarrow Y^*$ bounded. Given $E \subseteq X$ finite-dim and $\varepsilon > 0$ set $P_{(E,\varepsilon)} : E \longrightarrow W$ the operator described in (5). We would want, somehow, to glue the "local operators" $T \circ P_{(E,\varepsilon)}$.

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Let Y Banach space and $T : W \longrightarrow Y^*$ bounded. Given $E \subseteq X$ finite-dim and $\varepsilon > 0$ set $P_{(E,\varepsilon)} : E \longrightarrow W$ the operator described in (5). We would want, somehow, to glue the "local operators" $T \circ P_{(E,\varepsilon)}$. Define

$$T_{(E,\varepsilon)}(x) := \begin{cases} T(P_{(E,\varepsilon)})(x) & \text{if } x \in E, \\ 0 & \text{otherwise} \end{cases}$$

A Lindenstrauss compactness argument

Theorem (Definition of (C) locally complemented subspace)

Let X be a Banach space and W subspace. TFAE:

(1) *For every Banach space Y, every bounded linear operator $T \in L(W, Y^*)$ admits an extension $\tilde{T} \in L(X, Y^*)$ such that $\|\tilde{T}\| \leq C\|T\|$.*

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It is **not** linear but, somehow, it is "more and more linear" when E grows.

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$$T_{(E,\varepsilon)} \in \prod_{x \in X}((C+1)\|T\|\|x\|B_{Y^*}, w^*),$$

which is a compact topological space by Tychonoff theorem. Take $\tilde{T}$ a cluster point of the net $(T_{(E,\varepsilon)})_{(E,\varepsilon)\in\Gamma}$, and it satisfies the desired requirements.

Examples where projective tensor does not respect subspaces. Isomorphic case.

Corollary

Let X and Y be two Banach spaces and let $W \subseteq X$ be a subspace. The following assertions are equivalent:

1. $\|u\|_{W\widehat{\otimes}_\pi Y} \leq C\|u\|_{X\widehat{\otimes}_\pi Y}$ *for every* $u \in W \otimes Y$.
2. *Every bounded linear operator* $T \in L(W, Y^*)$ *admits an extension* $\tilde{T} \in L(X, Y^*)$ *such that* $\|\tilde{T}\| \leq C\|T\|$.

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We are able to find X, Y, W such that $W\widehat{\otimes}_\pi Y$ is not isomorphically a subspace of $X\widehat{\otimes}_\pi Y$.

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We are able to find X, Y, W such that $W\widehat{\otimes}_\pi Y$ is not isomorphically a subspace of $X\widehat{\otimes}_\pi Y$. Take a reflexive Banach space X not isomorphic to a Hilbert space. There exists a subspace $Y \subseteq X$ for which there is no projection $P : X \longrightarrow Y$. This means that the operator $i : Y \longrightarrow Y$ does not admit any continuous extension to $P : X \longrightarrow Y$.

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We are able to find X, Y, W such that $W\widehat{\otimes}_\pi Y$ is not isomorphically a subspace of $X\widehat{\otimes}_\pi Y$. Take a reflexive Banach space X not isomorphic to a Hilbert space. There exists a subspace $Y \subseteq X$ for which there is no projection $P : X \longrightarrow Y$. This means that the operator $i : Y \longrightarrow Y$ does not admit any continuous extension to $P : X \longrightarrow Y$. Hence $Y\widehat{\otimes}_\pi Y^*$ is **not** isomorphically a subspace of $X\widehat{\otimes}_\pi Y^*$.

Examples where projective tensor does not respect subspaces. Finite-dimensional case.

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Let X be an infinite-dimensional Banach space with the property that there exists $\lambda \geq 1$ so that for every finite-dimensional subspace E of X there exists a projection $P : X \longrightarrow E$ with $\|P\| \leq \lambda$. Then X is isomorphic to a Hilbert space and, indeed, it is $4\lambda^2$

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Take an infinite-dimensional Banach space X which is not isomorphic to a Hilbert space and let $\lambda > 1$. Then there exists a subspace $E \subseteq X$ such that, for every bounded projection $P : X \longrightarrow E$ it follows $\|P\| > \lambda$.

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If not, then for every $E \subseteq F \subseteq X$ there would exists a bounded projection $P_F : F \longrightarrow E$ with $\|P_F\| \leq \lambda$.

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If not, then for every $E \subseteq F \subseteq X$ there would exists a bounded projection $P_F : F \longrightarrow E$ with $\|P_F\| \leq \lambda$. Set $\Gamma := \{F \subseteq X : E \subseteq F, dim(F) < \infty\}$ ordered with the order inclusion. Define

$$T_F(x) := \begin{cases} P_F(x) & \text{if } x \in F, \\ 0 & \text{otherwise} \end{cases}$$

Observe $T_F \in \prod_{x \in X} (\lambda B_E, \| \cdot \|)$, which is compact.

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Observe $T_F \in \prod_{x \in X} (\lambda B_E, \|\cdot\|)$, which is compact. Take a cluster point in the above compact space, say P. It can be proved that $P : X \longrightarrow E$ is a linear projection and $\|P\| \leq \lambda$.

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Consequently, given the identity operator $i : E \longrightarrow E$, for any extension $P : F \longrightarrow E$ we get $\|P\| \geq \lambda$.

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Consequently, given the identity operator $i : E \longrightarrow E$, for any extension $P : F \longrightarrow E$ we get $\|P\| \geq \lambda$. This implies that

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We have proved the following result.

Theorem

Let X be an infinite-dimensional Banach space which is not isomorphic to a Hilbert space. Then, for every $\lambda > 0$ there are $E \subseteq F \subseteq X$ finite dimensional such that

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The principle of local reflexivity establishes X is 1-locally complemented in X^{**}. This implies that for any Banach space Y and every bounded operator $T : X \longrightarrow Y^*$ there exists a norm-extending extension $\hat{T} : X^{**} \longrightarrow Y^*$.

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If X and Y are Banach space then $X\widehat{\otimes}_\pi Y$ is isometrically a subspace of $X^{**}\widehat{\otimes}_\pi Y$.

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In general, if T and S are onto linear isomorphisms, then so is $T \otimes S$. For into isomorphisms the above result does not follow. However, the above operators will reveal a different better behaviour which justifies the name “projective” for the π-norm.

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Proposition

If $P : X \longrightarrow Z$ and $Q : Y \longrightarrow W$ is a projection, then $P \otimes Q : X\widehat{\otimes}_\pi Y \longrightarrow Z\widehat{\otimes}_\pi W$ is a projection.

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