Geometry of tensor products and bilinear mappings in Banach spaces I

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XXII Lluís Santaló School 2023 Linear and non-linear analysis in Banach spaces

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Departamento de Análisis Matemático

Support

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Junta de Andalucía

References

R. A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monographs in Mathematics, Springer-Verlag, London, 2002.

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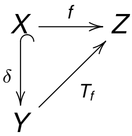

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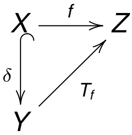

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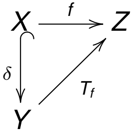

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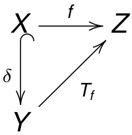

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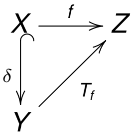

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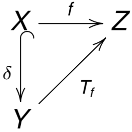

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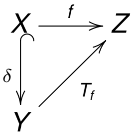

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Let us start with a vector spaces framework, and let us talk later about norms and continuity.

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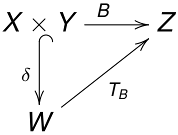

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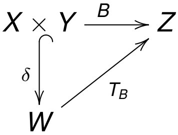

Where can we find W?

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$$\begin{array}{ccc} X \times Y & \xrightarrow{\;B\;} & \mathbb{K} \\ {\scriptstyle\delta}\big\downarrow & \nearrow_{T_B} & \\ W & & \end{array}$$

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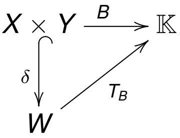

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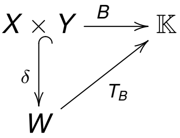

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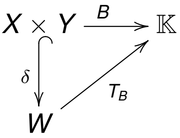

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This makes sensible to look for W inside $\mathrm{Bil}(X \times Y)^\sharp$, the algebraic dual of the vector space $\mathrm{Bil}(X \times Y)$.

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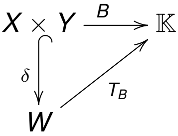

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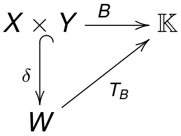

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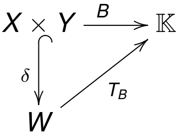

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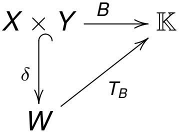

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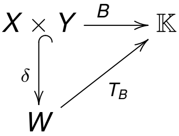

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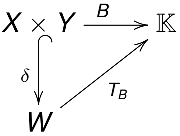

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Tensor product of vector spaces

The tensor product of two vector spaces X and Y is defined by

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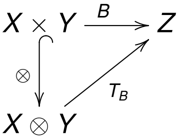

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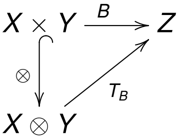

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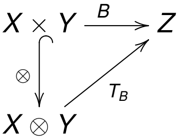

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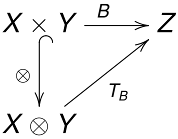

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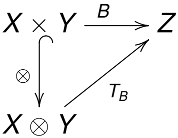

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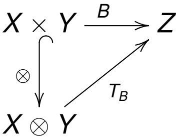

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It is **reasonable** to require $\|x \otimes y\| \leq \|x\|\|y\|$.

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$$\|B(x, y)\| \leq C\|x\|\|y\| \ \forall x \in X, y \in Y.$$

We denote by $\mathcal{B}(X \times Y, Z)$ the Banach space of all bilinear continuous mapping under the norm $\|B\| := \sup_{x \in B_X, y \in B_Y} \|B(x, y)\|$.

Bilinear continuous mappings and “reasonable” norms

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In view of the above reasonability is nothing but forcing the bilinear mapping $\otimes : X \times Y \longrightarrow X \otimes Y$ to be continuous.

The projective norm

$$\|x \otimes y\| \leq \|x\|\|y\|.$$

Now if $u = \sum_{i=1}^{n} x_i \otimes y_i$, then

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Given $u \in X \otimes Y$ we define

$$\|u\| = \|u\|_{\pi} = \inf \left\{ \sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i \right\}.$$

The projective norm, first properties

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Projective norm. Infinite-dimensional case

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$$(f \otimes g)(u) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x_n) g(y_n) = g\left(\sum_{n=1}^{\infty} \frac{1}{2^n} f(x_n) y_n\right).$$

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By the same reason, $(f \otimes g)(u) = g\left(\sum_{i=1}^{p} f(a_i) b_i\right)$.

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The arbitrariness of g implies $\sum_{n=1}^{\infty} \frac{1}{2^n} f(x_n) y_n = \sum_{i=1}^{p} f(a_i) b_i \ \forall f$. Evaluating at f_n we get $\frac{1}{2^n} y_n = \sum_{i=1}^{p} f_n(a_i) b_i \in span\{b_1, \ldots, b_p\}$, which is impossible since $\{y_n : n \in \mathbb{N}\}$ is linearly independent.

The projective tensor product

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Given two Banach spaces X and Y, the *projective tensor product of* X *and* Y, denoted by $X\widehat{\otimes}_\pi Y$, is defined as the completion of $(X \otimes Y, \|\cdot\|_\pi)$.

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There are more norm which can be defined on $X \otimes Y$, like the *injective norm*, which is defined as

$$\left\|\sum_{i=1}^n x_i \otimes y_i\right\|_\varepsilon = \sup\left\{\sum_{i=1}^n x^*(x_i)y^*(y_i) : x^* \in B_{X^*}, y^* \in B_{Y^*}\right\}.$$

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This norm satisfies $\|u\|_\varepsilon \leq \|u\|_\pi$, and this is the operator norm when we view $u = \sum_{i=1}^n x_i \otimes y_i$ as an operator

$$\begin{array}{ccc} X^* & \longrightarrow & Y \\ x^* & \longmapsto & \sum_{i=1}^n x^*(x_i)y_i. \end{array}$$

The dual of $X\widehat{\otimes}_\pi Y$ (I)

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- Taking inf. in repr. of $u \Rightarrow |T_B(u)| \leq \|B\|\|u\|_\pi$. This proves that T_B acts linear and continuously on $X \otimes Y$ (hence on $X\widehat{\otimes}_\pi Y$) and $\|T_B\|_\pi \leq \|B\|$.

The dual of $X\widehat{\otimes}_\pi Y$ (II)

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Finally, given $\varphi \in (X\widehat{\otimes}_\pi Y)^*$, set $B(x,y) := \varphi(x \otimes y)$, we get $B \in \mathcal{B}(X \times Y)$

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Finally, given $\varphi \in (X\widehat{\otimes}_\pi Y)^*$, set $B(x,y) := \varphi(x \otimes y)$, we get $B \in \mathcal{B}(X \times Y)$ and $T_B - \varphi = 0$ on $X \otimes Y$, so $T_B = \varphi$.

The dual of $X\widehat{\otimes}_\pi Y$ (III)

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Theorem

The mapping

$$\begin{array}{ccc} \mathcal{B}(X \times Y) & \longrightarrow & (X\widehat{\otimes}_\pi Y)^* \\ B & \longmapsto & T_B \end{array}$$

is an onto linear isometry

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Moreover, the natural onto linear isometry

$$\begin{array}{ccc} \mathcal{B}(X\times Y) & \longrightarrow & L(X,Y^*) \\ B & \longrightarrow & T(x):=B(x,\cdot) \end{array}$$

makes in practice to write $(X\widehat{\otimes}_\pi Y)^* = L(X,Y^*) = B(X\times Y)$.

The linearisation property

Another consequence of the above ideas is that the mapping

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establishes an onto linear isometry (and more often than not, the identification is replaced with an “equality”). Moreover, this gives the desired linearisation property of projective tensor products.

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Corollary

Let X and Y be two Banach spaces. Then

$$B_{X\widehat{\otimes}_\pi Y} = \overline{\text{conv}}(B_X \otimes B_Y).$$

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It is equivalent to proving that $B_X \otimes B_Y$ is norming for $(X\widehat{\otimes}_\pi Y)^*$.

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It remains a standard application of Hahn-Banach theorem.