# Distances between $\mathcal{C}(K)$ spaces

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and the distances are defined to be $\infty$ if the spaces $E_1, E_2$ are not linearly isomorphic, Lipschitz isomorphic, or uniformly homeomorphic, respectively. If $T$ is a bounded linear operator, then $Lip(T) = \|T\|$ and if $T$ is Lipschitz, then $l_\infty(T) \leq Lip(T)$, and thus

$$d_u(E_1, E_2) \leq d_{Lip}(E_1, E_2) \leq d_{BM}(E_1, E_2).$$

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(ii) A Banach space $E$ is Lipschitz isomorphic to $\mathcal{C}([0,\omega])$ iff it linearly isomorphic to it (Godefroy, Calton, Lancien, 2000), that is,

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- The problem of determining which compact spaces $L$ can replace $[0,\omega]$ in the above results is wide open.

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- The only values of the Banach-Mazur distance that are known to be attained between $C(K)$ spaces are $1, 2$ and $3$.
- Further, whenever a $C(K)$ space is isomorphic to $c_0$, then $d_{BM}(C(K), c_0)$ is an odd integer, and for each odd natural number $m$ other than 1 there exists a $C(K)$ space such that $d_{BM}(c_0, C(K)) = m$ (Candido, 2018).

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**Problem (probably due to Pelczynski)**

*Is the Banach-Mazur distance between two isomorphic $\mathcal{C}(K)$ spaces always an integer?*

# The linear distance 2

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- $K_1, K_2$ are homeomorphic $\iff d_{BM}(\mathcal{C}(K_1), \mathcal{C}(K_2)) < 2$ (independently Amir, 1965 and Cambern, 1966). Thus, there are no compact spaces $K_1$, $K_2$ such that $1 < d_{BM}(\mathcal{C}(K_1), \mathcal{C}(K_2)) < 2$.

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- (Cohen, 1975) and (Chu, Cohen, 1995) provided examples of pairs of $\mathcal{C}(K)$ spaces where the distance 2 is attained.

# The nonlinear distance 2

- Several authors have been working on nonlinear versions of the Amir-Cambern theorem, in the sense that they proved results saying that if a certain nonlinear distance between $\mathcal{C}(K)$ spaces is small, then the underlying compact spaces must be homeomorphic. There were results in this direction proved by (Jarosz, 1989), (Dutrieux and Kalton, 2005), (Górak, 2011), and (Galego and Porto da Silva, 2016).

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- The strongest and most general result so far was proved in (Galego and Porto da Silva, 2016), which in particular proves the equivalence

$$K_1, K_2 \text{ are homeomorphic } \iff d_u(\mathcal{C}(K_1), \mathcal{C}(K_2)) < 2$$

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  valid for all compact spaces $K_1, K_2$ (recall that this constant is optimal even in the linear case).
- It follows that the spaces $\mathcal{C}(K_1), \mathcal{C}(K_2)$ mentioned in the previous slide satisfy

$$d_u(\mathcal{C}(K_1), \mathcal{C}(K_2)) = d_{Lip}(\mathcal{C}(K_1), \mathcal{C}(K_2)) = d_{BM}(\mathcal{C}(K_1), \mathcal{C}(K_2)) = 2.$$

# Derivatives

In the rest of the talk, only the Banach-Mazur distance will be considered (because there is nothing in this direction known about nonlinear distances).

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If $K$ is compact and $\alpha$ an ordinal, then the *Cantor-Bendixon derivative* of $K$ of order $\alpha$ is

$$
\begin{aligned}
K^{(0)} &= K,\\
K^{(1)} &= \{x \in K : x \text{ is an accumulation point of } K\},\\
K^{(\alpha)} &= (K^{\beta})^{(1)}, \quad \alpha = \beta + 1,\\
K^{(\alpha)} &= \bigcap_{\beta<\alpha} K^{(\beta)}, \quad \alpha \text{ limit.}
\end{aligned}
$$

$K$ is *scattered* if there exists $\alpha$ such that $K^{(\alpha)} = \emptyset$. In that case, the smallest ordinal $\alpha$ such that $K^{(\alpha)} = \emptyset$ is called *height* of $K$, denoted by $ht(K)$ (and it is a successor ordinal).

# The linear distance 3

- If there exists an ordinal $\alpha$ such that $\left|K_1^{(\alpha)}\right| \neq \left|K_2^{(\alpha)}\right|$, then $d_{BM}(\mathcal{C}(K_1), \mathcal{C}(K_2)) \geq 3$ (Gordon, 1970).

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- This applies in particular in the following cases:
  (i) $K_1, K_2$ are scattered compact spaces of different heights,
  (ii) $K_1, K_2$ are nonhomeomorphic countable compact spaces (because the homeomorphism class of each countable compact space is uniquely determined by its height and the cardinality of the highest nonempty derivative (Mazurkiewicz, Sierpinski, 1920)).

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- It holds $d_{BM}(\mathcal{C}([0, \omega]), \mathcal{C}([0, \omega 2])) = 3$ (Gordon, 1970).

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**Proof.**

Since $\left|[0,\omega]^{(1)}\right| = |\{\omega\}| = 1$ and $\left|[0,\omega 2]^{(1)}\right| = |\{\omega,\omega 2\}| = 2$, $d_{BM}(\mathcal{C}([0,\omega]),\mathcal{C}([0,\omega 2])) \geq 3$ by the theorem of Gordon.
On the other hand, it is elementary to check that the mapping $T : \mathcal{C}([0,\omega 2]) \to \mathcal{C}([0,\omega])$ given by

$$
\begin{aligned}
Tf(1) &= f(\omega) - f(\omega 2) \\
Tf(2m) &= f(m) - \frac{1}{2}(f(\omega) - f(\omega 2)), \quad m \in \mathbb{N}, \\
Tf(2m+1) &= f(\omega + m) + \frac{1}{2}(f(\omega) - f(\omega 2)), \quad m \in \mathbb{N}, \\
Tf(\omega) &= \frac{1}{2}(f(\omega) - f(\omega 2))
\end{aligned}
$$

is a surjective isomorphism with $\|T\| = 2$ and $\left\|T^{-1}\right\| = \frac{3}{2}$. □

# Distances from $\mathcal{C}([0,\omega])$

- (Candido, Galego, 2013) estimated the distance of $\mathcal{C}([0,\omega])$ from the other $\mathcal{C}(K)$ spaces isomorphic to it with an error of at most 2. Some of their methods were improved in unpublished notes of Cuth and his student Havelka. The way to find the upper estimates is achieved by using regular matrices to construct certain isomorphisms between the respective spaces (the isomorphism from the previous slide corresponds in this setting to the matrix $\begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$).

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- It is well-known that a $\mathcal{C}(K)$ space is isomorphic to $\mathcal{C}([0,\omega])$ iff $K$ is homeomorphic to $[0,\omega^\alpha m]$ for some $\alpha, m \in \mathbb{N}$ (Bessaga, Pelczynski, 1960).

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If $m>3$, then

$$3\leq d_{BM}(\mathcal{C}([0,\omega]),\mathcal{C}([0,\omega m]))\leq 2+\sqrt{5} \quad \text{(Candido, Galego, 2013)}.$$

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If $\alpha\in\mathbb{N}$, then

$$2\alpha-1\leq d_{BM}(\mathcal{C}([0,\omega]),\mathcal{C}([0,\omega^\alpha]))\leq \alpha+\sqrt{(\alpha-1)(\alpha+3)}, \text{ and}$$

$$2\alpha+1\leq d_{BM}(\mathcal{C}([0,\omega]),\mathcal{C}([0,\omega^\alpha m]))\leq \alpha+1+\sqrt{(\alpha(\alpha+4))}$$

if $m>1$.(Candido, Galego, 2013).

# Lower estimates of the distance based on the Cantor-Bendixon derivatives

- The lower bounds from the previous slide were generalized to the context of general scattered spaces in (R., 2021) and (R., Somaglia, 2022) (here the distance is estimated from below based on the heights of the compact spaces), and

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- in a recent preprint (R., 2023) further generalized to spaces that need not be scattered (here the distance is estimated from below based on a certain cardinal invariant of Cantor-Bendixon derivatives of the compact spaces).

# Acknowledgement

Thank you.