Different convexity concepts in norms of Banach spaces

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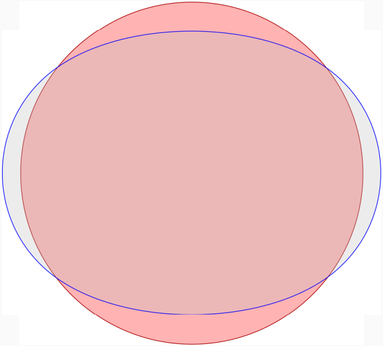

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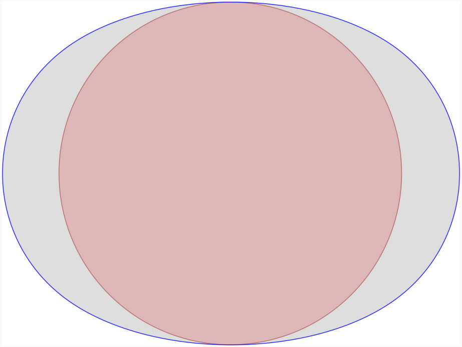

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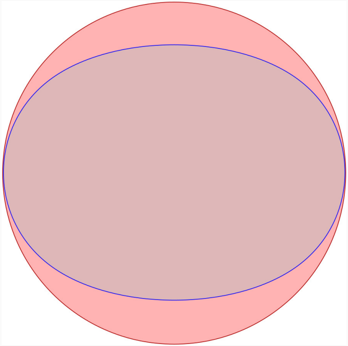

Rotundity vs Uniform Rotundity of norms

Rotundity (or strict convexity)

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Uniform rotundity is a very strong property: A Banach space admits a UR equivalent norm if and only if it is superreflexive.

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Local Uniform Rotundity

$\|\cdot\|$ is locally uniformly rotund (LUR) if for any $x_0 \in S_X$ and any sequence $(y_n)_n \subset S_X$,

$$\left\|\frac{x_0 + y_n}{2}\right\| \to 1 \text{ implies that } \|x_0 - y_n\| \to 0$$

Relaxing UR: Directions

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Uniform Rotundity in Every Direction

$\|\cdot\|$ is uniformly rotund in every direction (URED) if for any $v \in S_X$ and any $(x_n)_n, (y_n)_n \subset S_X$ with $y_n - x_n \in \operatorname{span}\{v\}$,

$$\left\| \frac{x_n + y_n}{2} \right\| \to 1 \text{ implies that } \|x_n - y_n\| \to 0$$

Distinguishing Rotundity concepts

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Can we construct in every separable Banach space a norm which is LUR but not URED?

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Theorem (Hájek, Q. 2023)

The answer is yes to all three questions. In fact, such norms are dense whenever they exist.

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Start by constructing a sphere which contains segments, but only in a particular set of chosen directions.

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An idea:

Start by constructing a sphere which contains segments, but only in a set of chosen directions.

Q: How do we get segments in the sphere in only a particular direction?

Thank you!