On the numerical index of the real two-dimensional L_p space

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joint work with J. Merí

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Joint work

J. Merí and A. Quero
On the numerical index of absolute symmetric norms on the plane
Linear Mult. Algebra **69** (2021), no. 5, 971–979.

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Introduction

Definitions

Numerical radius (Bauer, Lumer, early 60's)

X Banach space, $T \in \mathcal{L}(X)$

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- $0 \leqslant n(X) \leqslant 1$
- v and $\|\cdot\|$ are equivalent norms $\iff n(X) > 0$

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- $n(C(K)) = n(L_1(\mu)) = 1$

  $n(c_0) = n(\ell_1) = n(\ell_\infty) = 1$

  (Duncan–McGregor–Pryce–White, 1970)

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Problem in the real two-dimensional case

Question

Let $1 < p < \infty$, $p \neq 2$

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Recent results on numerical index of ℓ_p^2

Theorem (Merí–Q., 2021)

Let $p \in \left[\frac{3}{2}, 3\right]$. Then, $n(\ell_p^2) = M_p = \max_{t\in[0,1]} \dfrac{|t^{p-1} - t|}{1 + t^p}$.

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Theorem (Monika–Zheng, 2022)

Let $p \in [1 + \alpha_0, \alpha_1]$, where $\alpha_0 \approx 0.4547$, $\alpha_1 \approx 3.1993$. Then,
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Theorem (Merí–Q., 2023)

Let $p \in \left[\frac{6}{5}, \frac{3}{2}\right] \cup [3, 6]$. Then, $n(\ell_p^2) = M_p = \max_{t\in[0,1]} \dfrac{|t^{p-1} - t|}{1 + t^p}$.

Absolute symmetric norms on the plane

Absolute and symmetric norms

Definition

$\|\cdot\|\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ is absolute if

- $\|(1,0)\| = \|(0,1)\| = 1$
- $\|(a,b)\| = \|(|a|,|b|)\|$ for every $a, b \in \mathbb{R}$

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Examples

- ℓ_p-norms on $\mathbb{R}^2$
- Octogonal norms
- Norms whose unit ball is a regular polygon with $4n$ vertices, for $n \in \mathbb{N}$

Octogonal norms

Theorem (Martín, Merí, 2007)

For $\xi \in [0,1]$, $X_\xi = (\mathbb{R}^2, \|\cdot\|_\xi)$, where

$$\|(x,y)\|_\xi = \max\left\{|x|,\ |y|,\ \frac{|x|+|y|}{1+\xi}\right\}$$

for every $(x,y) \in \mathbb{R}^2$. Then

$$n(X_\xi) = \max\left\{\xi,\ \frac{1-\xi}{1+\xi}\right\} = v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

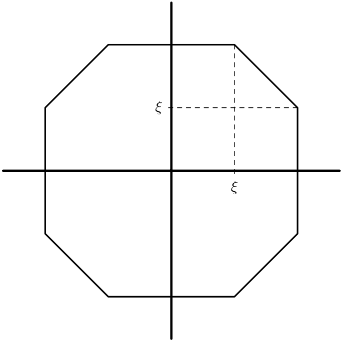

Figure: Octogonal norm

Regular polygons

Theorem (Martín, Merí, 2007)

Let $n \in \mathbb{N}$, X_n is the two dimensional real Banach space such that

$$\mathrm{ext}(B_{X_n}) = \{x_k : k = 1, 2, \dots, 4n\}$$

where

$$x_k = \left(\cos\left(\frac{k\pi}{2n}\right),\ \sin\left(\frac{k\pi}{2n}\right)\right)$$

for $k = 1, 2, \dots, 4n$. Then

$$n(X_n) = \tan\left(\frac{\pi}{4n}\right) = v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

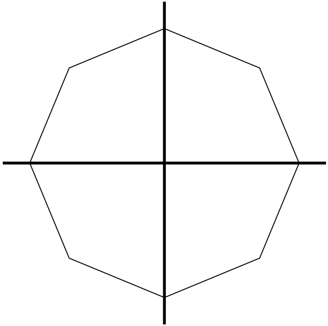

Figure: Regular polygon $n = 2$

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- Take $x_0\in S_X$ and $x_0^*\in S_{X^*}$ be such that $x_0^*(x_0)=1$ and $|x_0^*(I_4x_0)|=v(I_4)$

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Theorem (Merí–Q., 2021)

Let X be $\mathbb{R}^2$ endowed with an absolute and symmetric norm.
Let $x_0 \in S_X$ and $x_0^* \in S_{X^*}$ be such that $x_0^*(x_0) = 1$ and $|x_0^*(I_4 x_0)| = v(I_4)$ and write $c_j = |x_0^*(I_j x_0)|$ for every $j = 1, \ldots, 4$.

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If $c_4 = 0$, then $n(X) = 0$. If otherwise $c_4 > 0$, then

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Moreover, if the inequality $c_4 \left(1 + \frac{1}{c_2} + \frac{1}{c_3}\right) \leqslant 1$ holds, then

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- $$v(T) = \max\left\{\max_{t\in[0,1]} \frac{|a + d\,t^p| + |b\,t + c\,t^{p-1}|}{1+t^p}, \max_{t\in[0,1]} \frac{|d + a\,t^p| + |c\,t + b\,t^{p-1}|}{1+t^p}\right\}$$

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- $\|T\| \leqslant \|T\|_1^{1/p} \|T\|_\infty^{1/q} \leqslant \max\{\|T\|_1, \|T\|_\infty\}$ (Riesz-Thorin theorem)
- $$v(T) = \max\left\{ \max_{t\in[0,1]} \frac{|a + d\,t^p| + |b\,t + c\,t^{p-1}|}{1+t^p}, \max_{t\in[0,1]} \frac{|d + a\,t^p| + |c\,t + b\,t^{p-1}|}{1+t^p} \right\}$$
- Fix $t_0 \in]0,1[$ such that $M_p = \max_{t\in[0,1]} \frac{|t^{p-1} - t|}{1+t^p} = \frac{|t_0^{p-1} - t_0|}{1+t_0^p}$

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- $\frac{v(T)}{\|T\|_1^{1/p} \|T\|_\infty^{1/q}} \geqslant \frac{|t_0^{p-1} - t_0|}{1+t_0^p}$ for every $T \in \mathcal{L}(\ell_p^2)$ and $p \in \left]1, \frac{3}{2}\right] \cup [3, \infty[$?

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Theorem (Merí-Q., 2022)

Let $p \in \left[\frac{6}{5}, \frac{3}{2}\right] \cup [3, 6]$. Then, $n(\ell_p^2) = M_p$.

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