On the weak topology in Lipschitz free spaces

Colin PETITJEAN

Lluís Santaló School 2023
Santander, Spain

Université Gustave Eiffel

LAMA LABORATOIRE D'ANALYSE ET DE MATHÉMATIQUES APPLIQUÉES — UMR 8050

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Part 2

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- [Godefroy – Kalton, 2003]: If X is a separable Banach space, then X is isometric to a subspace of $\mathcal{F}(X)$.

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Question

$$\mathcal{F}(\mathbb{R}^2) \simeq \mathcal{F}(\mathbb{R}^3)?$$

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→ A (probably similar) proof is due to [Aliaga – Pernecká – Smith, ????]

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- [Cabrera-Padilla – Jiménez-Vargas, 2016]: $\widehat{f} : \mathcal{F}(M) \to \mathcal{F}(N)$ is (weakly) compact if and only if

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Remark **[Abbar – Coine – P., 2023]**:
Also works for weighted versions $w\widehat{f}$ of these "Lipschitz operators".

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- $S \subset \mathcal{FS}_2(M)$. $\square$

Remark **[Abbar – Coine – P., 2023]**:
Also works for weighted versions $w\widehat{f}$ of these "Lipschitz operators".

$\to$ Leads to characterizations (in terms of metric properties of f) of those (weighted) Lipschitz operators $w\widehat{f}$ which are (weakly) compact.

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Theorem (Kalton, 2004)

If M is bounded and $0 < p < 1$ then $\mathcal{F}(M, d^p)$ has the Schur property.

Theorem (Kalton, 2004 – Upgraded version)

If $\mathrm{lip}_0(M)$ *separates the points of* M *uniformly, then* $\mathcal{F}(M)$ *has the Schur property.*

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Summary: Weak precompactness $\implies$ Kalton's property $\iff$ Tightness.

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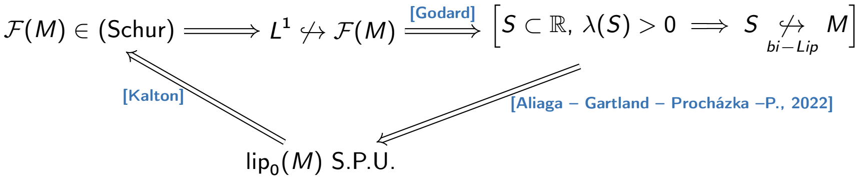

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Therefore for general M:

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Remark: Thanks to [Godefroy – Kalton, 2003], c_0 is isometric to a subspace of $\mathcal{F}(c_0)$, and therefore $\mathcal{F}(c_0)$ is not w.s.c.

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- $\mathcal{F}(S)$ is w.s.c. for every superreflexive space S.
- For every $p \in (1, \infty)$, $\mathcal{F}(\ell_p) \not\simeq \mathcal{F}(c_0)$.

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To be continued...

Muchas gracias por su atención!