Dvoretzky-type theorem for locally finite subsets of a Hilbert space

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  - A bilipschitz embedding whose distortion does not exceed $C\in[1,\infty)$ is called *C-bilipschitz*.
- **Theorem 1 (joint work with Florin Catrina and Sofiya Ostrovska)**: Given any $\varepsilon>0$, every locally finite subset of ℓ_2 admits a $(1+\varepsilon)$-bilipschitz embedding into an arbitrary infinite-dimensional Banach space.

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- (1) *Asymptotic theory of finite-dimensional Banach spaces*, also called *Local Theory* if the goal is to apply it to study infinite-dimensional Banach spaces.
- (2) The theory of *Metric Embeddings*, or, more precisely, its part devoted to embeddings of discrete metric spaces into Banach spaces.

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- The stated result is at the intersection of two very important parts of Banach space theory.
- (1) *Asymptotic theory of finite-dimensional Banach spaces*, also called *Local Theory* if the goal is to apply it to study infinite-dimensional Banach spaces.
- (2) The theory of *Metric Embeddings*, or, more precisely, its part devoted to embeddings of discrete metric spaces into Banach spaces.
- One of the general directions of the local theory is to understand, to what extent the structure of the general infinite-dimensional Banach space resembles the structure of the Hilbert space.

- The first significant success in this direction was the result of Dvoretzky and Rogers (1950). By showing that the structure of the unit ball of any high-dimensional Banach space has significant similarities with the ball of a Hilbert space they proved that for any sequence $\{a_i\}$ of positive numbers satisfying $\sum_{i=1}^{\infty} a_i^2 < \infty$ any infinite-dimensional Banach space contains a series $\sum_{i=1}^{\infty} x_i$ which is unconditionally convergent ($=$ converges after any rearrangement) and satisfies $\|x_i\| = a_i$.

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- Grothendieck (1953) analyzed this result of Dvoretzky and Rogers, and (among other things) conjectured the following result, somewhat later proved by Dvoretzky (announcement - 1959, publication - 1961).

- **Dvoretzky Theorem:** Let $k \in \mathbb{N}$, $k \geq 2$, and $0 < \varepsilon < 1$. There exists $N = N(k, \varepsilon) \in \mathbb{N}$ so that every normed space having more than N dimensions - in particular every infinite-dimensional normed space - has a k-dimensional subspace whose Banach-Mazur distance from the k-dimensional Hilbert space is less than $(1 + \varepsilon)$.

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- If you never saw this result, the following restatement can help to understand the condition on the Banach-Mazur distance. It means that there exists a linear map T from a k-dimensional Euclidean space ℓ_2^k (standard notation) into X, for which

$$\|x\| \leq \|Tx\| \leq (1 + \varepsilon)\|x\|.$$

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- Slightly later I shall mention some Dvoretzky-type theorems and related open problems. Now I would like to remind some facts about Metric Embeddings.

- The first step in the study of Metric Embeddings were made almost simultaneously with the creation of the theory of metric spaces, namely Frèchet (1910) proved that each separable metric space embeds isometrically into ℓ_∞ and each n-point metric space embeds isometrically into ℓ_∞^{n-1}.

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- The development of the theory of Metric Embeddings started to accelerate in the first half of 80s, with the following results proved by Assouad (1983), Johnson-Lindenstrauss (1984), and Bourgain (1985).

- **Assouad (1983):** If a metric space $(\mathcal{M}, d_{\mathcal{M}})$ satisfies the condition: there exists $K < \infty$ such that each ball in $\mathcal{M}$ is covered by at most K balls of of twice smaller diameter, then for every $p \in (0,1)$ there exists $1 \le C(p,K) < \infty$ and $N(p,K) \in \mathbb{N}$ such that $(\mathcal{M}, d^p_{\mathcal{M}})$ admits a bilipschitz embedding into the Euclidean space $\mathbb{R}^{N(p,K)}$ with distortion at most $C(p,K)$.

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- **Johnson-Lindenstrauss (1984):** For each $\varepsilon > 0$ there exists $K(\varepsilon) < \infty$ such that for each n-element metric space $(\mathcal{M}, d_{\mathcal{M}})$ in ℓ_2, there exists an embedding of $(\mathcal{M}, d_{\mathcal{M}})$ into the Euclidean space $\mathbb{R}^{K(\varepsilon)\log_2 n}$ with distortion $\le (1+\varepsilon)$.

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- **Bourgain (1985):** For each n-element metric space $(\mathcal{M}, d_{\mathcal{M}})$ there exists an embedding into ℓ_2 with distortion $\leq C \log_2 n$, where C is an absolute constant. (Later it was shown that up to the value of C this result is the best possible.)

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- In the 1990s, two important ideas on applications of embeddings of discrete metric spaces into Banach spaces emerged:

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- For Gromov's approach it is sufficient to find embeddings which are weaker than bilipschitz: A map $F : \mathcal{M} \to \mathcal{L}$ is called a *coarse embedding* if there exist nondecreasing, infinite at ∞ functions $\rho^-, \rho^+ : [0, \infty) \to [0, \infty)$ so that for all $u, v \in \mathcal{M}$

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- The metric spaces in Gromov's problems are infinite groups G generated by finite sets S, regarded as vertex sets of infinite graphs in which two vertices $u, v \in G$ are joined by an edge if and only if $u = vs$ for some $s \in S$ (the metric $d(u, v) =$ the minimal number of multiplication of v by elements of S after which we get u).

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- There was a significant progress along the lines suggested by Gromov; an (early) summary of it can be found in the International Congress talk of Guoliang Yu (2006).

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- Let me describe the general idea of this. There are many computer science problems on finite metric spaces $\mathcal{M}$. Among these problems there are problems for which fast algorithms are known if $\mathcal{M}$ is a subset of ℓ_1 with the induced metric, but it is believed that there are no such algorithms in general (the corresponding problems are NP-complete, the standard term in the theory of Algorithms).

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- In many such situations, if the metric space $\mathcal{M}$ is not isometric to a subset of ℓ_1, but admits a low-distortion embedding into ℓ_1, we can use the mentioned fast algorithm for subsets of ℓ_1, and get a fast useful approximation of the solution for $\mathcal{M}$.

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- See International Congress (ICM) talks by Linial (2002) and Naor (2010, 2018) for description of the progress in this direction. There are also books on Approximation Algorithms.

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- First, we recall the open problem on the validity of a *finite isometric Dvoretzky theorem* for all infinite-dimensional Banach spaces.
- **Problem (M.O., 2015):** Do there exist a finite subset F of ℓ_2 and an infinite-dimensional Banach space X such that F does not admit an isometric embedding into X?

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- **Coming closer to Theorem 1 (M.O. 2009):** Each locally finite subset of ℓ_2 admits a bilipschitz embedding into arbitrary infinite-dimensional Banach space.
- In 2009 I did not try to give an estimate for the distortion, it is some number below 100, but not far from it. Theorem 1 of this talk is the best possible result in this direction.

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- *Let X be a Banach space containing a subspace with an unconditional basis which does not contain ℓ_∞^n uniformly. Then ℓ_2 embeds coarsely into X.*
- It was natural to check whether one can get the following common generalization (of the mentioned results): *Is it true that ℓ_2 embeds coarsely into an arbitrary infinite-dimensional Banach space?*

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- (Repeated from the previous slide) *Is it true that ℓ_2 embeds coarsely into an arbitrary infinite-dimensional Banach space?*
- This problem was posed in M.O.(2006) and repeated in M.O. (2009). A positive answer to this problem would be a very impressive Dvoretzky type theorem.

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- (Repeated from the previous slide) *Is it true that ℓ_2 embeds coarsely into an arbitrary infinite-dimensional Banach space?*
- This problem was posed in M.O.(2006) and repeated in M.O. (2009). A positive answer to this problem would be a very impressive Dvoretzky type theorem.
- Yet, as the matter stands, it was answered in the negative by Baudier-Lancien-Schlumprecht (2018) using a very elegant argument. A typical counterexample is the space constructed by Tsirelson (1974).

- One of the most important directions related to the Dvoretzky theorem is finding optimal estimates for the function $N(k, \varepsilon)$ in its statement. Many aspects of this problem have been investigated staring with Milman (1971).

Other directions of research related to Dvoretzky Theorem

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- Starting with the paper of Bourgain-Figiel-Milman (1986), a parallel theory for metric spaces was developed. In this theory the main goal is estimating from below the size - defined either as cardinality or in some measure-theoretic ways - of subsets of a metric space which admit low-distortion embeddings into a Hilbert space. The theory became very active after the fundamental paper Bartal-Linial-Mendel-Naor (2005). One can find a short survey in Section 8 of Naor's paper on Ribe program (2012).

Discussion of the proof of Theorem 1

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- Dvoretzky theorem implies that in any X there are arbitrarily large subspaces which are arbitrarily close to Euclidean spaces.
- Using Mazur's techniques for constructing basic sequences one can organize such almost-Euclidean spaces into a rather decent finite-dimensional Schauder decomposition (FDD).
- "Rather decent" here means that we can require FDD projections to have norms close to 1. (We can require even a bit more.)

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$$F_1 \subset F_2 \subset F_3 \subset \cdots \subset F_n \subset \dots .$$

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- A natural idea is to find in X and FDD of the form

$$F_1 \oplus F_2 \oplus F_3 \oplus \cdots \oplus F_n \oplus \dots.$$

- The space X does not have to contain such FDD isometrically, but it contains such FDD up to a linear map of an arbitrarily small distortion.

What is next?

- Now we can split $\mathcal{M}$ into annuli $\mathcal{A}_i := \{x \in \mathcal{M} : \ \rho_{i-1} \le d(x,0) \le \rho_i\}$ where $\rho_0 = 0$ and $\{\rho_i\}_{i=0}^{\infty}$ is a so rapidly increasing sequence of positive numbers that if the embedding will (almost) preserve the norm of elements, to compute the distortion it can be enough to consider pairs x, y which are either in the same annulus, or in neighboring annuli.

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- We can try to map $\mathcal{A}_{2i-1}$ into F_i by the natural isometric embedding (F_i is spanned by a set containing $\mathcal{A}_{2i-1}$). After that we can try to "bend" the complementary (even-numbered) annuli $\mathcal{A}_{2i}$ "between" F_i and F_{i+1} in the direct sum $F_i \oplus F_{i+1}$.

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- The problem is that we need "bending" with distortion arbitrarily close to 1. It is not an easy task because in an arbitrary Banach space X we do not have control (on close-to-isometric level) over the direct sums $F_i \oplus F_{i+1}$.

Bendings - let us defined them

- Let X and Y be Banach spaces such that there exist two linear isometric embeddings $I_1 : Y \to X$ and $I_2 : Y \to X$ with distinct images $Y_1 = I_1(Y)$ and $Y_2 = I_2(Y)$.

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- **Definition:** Let $C \in [1, \infty)$. A mapping $T : Y \to X$ is called a C*-bending* of Y in the space X from I_1 to I_2, with parameters (r, R), $0 < r < R < \infty$, if it is a C-bilipschitz embedding such that the restriction of T to the ball of radius r coincides with I_1 and the restriction of T to the exterior of the ball of radius R in Y coincides with I_2.

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- **Definition:** Let $C \in [1, \infty)$. A mapping $T : Y \to X$ is called a *C-bending* of Y in the space X from I_1 to I_2, with parameters (r, R), $0 < r < R < \infty$, if it is a C-bilipschitz embedding such that the restriction of T to the ball of radius r coincides with I_1 and the restriction of T to the exterior of the ball of radius R in Y coincides with I_2.
- When we consider a C-bending of Y in the space $X = Y \oplus Y$, we restrict our attention to the case where $I_1(y) = (y, 0)$ and $I_2(y) = (0, y)$ and call such bending *a C-bending of Y in the space $X = Y \oplus Y$ with parameters* (r, R), $0 < r < R < \infty$.

First steps in study bendings

- It would be very handy to have a result on existence of $(1+\varepsilon)$-bending with some parameters (r, R), $0 < r < R < \infty$, for every direct sum of the form $Y \oplus Y$ with direct sum projections having norms one. With such a result we could continue the argument on $F_1 \oplus F_2 \oplus F_3 \oplus \cdots \oplus F_n \oplus \dots$ which we started above.

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- However, such result does not hold.
- **Theorem 2 (COO):** There exists a 4-dimensional Banach space X satisfying the conditions:
  - **(A)** It is a direct sum of two 2-dimensional Euclidean spaces Y_1 and Y_2 with direct sum projections having norm 1.
  - **(B)** There exists $\varepsilon > 0$ such that for any (r, R) satisfying $0 < r < R < \infty$ and any isometric embeddings $I_1 : \ell_2^2 \to Y_1$ and $I_2 : \ell_2^2 \to Y_2$, there is no $(1+\varepsilon)$-bending of ℓ_2^2 in X from I_1 to I_2.

- Conclusion: We need to do more work on the FDD $F_1 \oplus F_2 \oplus F_3 \oplus \cdots \oplus F_n \oplus \dots$ to achieve $(1+\varepsilon)$ bending for an arbitrarily small $\varepsilon > 0$.

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- **Theorem 3 (Bending in 1-unconditional sums, COO):** Let Y be a finite-dimensional Banach space, and let $Z = (\mathbb{R}^2, \|\cdot\|_Z)$ for which the unit vector basis is 1-unconditional and normalized. Then for every $\varepsilon > 0$ and every pair (r, R) of positive numbers satisfying the condition
$$\frac{\varepsilon}{c_Z} \ln\left(\frac{R}{r}\right) = \frac{\pi}{2}, \tag{1}$$
there is a $\left(\frac{1+\varepsilon}{1-\varepsilon}\right)$-bending T of Y into the sum $X = Y \oplus_Z Y$ with parameters (r, R). Furthermore, the bending T satisfies
$$\|Tx\| = \|x\|, \tag{2}$$
and
$$(1-\varepsilon)\|x-y\| \le \|Tx - Ty\| \le (1+\varepsilon)\|x-y\|. \tag{3}$$

- The formula for bending is rather complicated. The main idea of it is to use the logarithmic spiral, that is a spiral in the plane which establishes a $(1+\varepsilon)$-bilipschitz embedding of the $(0,\infty)$ into the plane:

$$t \mapsto t(\cos(\varepsilon \ln t),\ \ \sin(\varepsilon \ln t)).$$

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- So, to complete the construction, we need to find an FDD $F_1 \oplus F_2 \oplus F_3 \oplus \dots \oplus F_n \oplus \dots$, for which sums of neighbors are unconditional or very close to unconditional.

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- So, to complete the construction, we need to find an FDD $F_1 \oplus F_2 \oplus F_3 \oplus \cdots \oplus F_n \oplus \dots$, for which sums of neighbors are unconditional or very close to unconditional.
- Note that such FDD does not have to be an unconditional FDD (so it can exist for a subspace of an arbitrary infinite-dimensional Banach space).

Unconditional sub-sums

- **Theorem 4 (Unconditionality for Sums of Euclidean Spaces, COO):** Given $n \in \mathbb{N}$, $\varepsilon \in (0,1)$, and $A \in [1,\infty)$ there exists $N \in \mathbb{N}$, such that, for every direct sum $X = X_1 \oplus X_2$ with both X_1 and X_2 isometric to ℓ_2^N, and the direct sum projections having norms $\leq A$, there are n-dimensional subspaces $Y_1 \subset X_1$ and $Y_2 \subset X_2$, such that the sum $Y_1 \oplus Y_2$ with the norm induced from X is $(1+\varepsilon)$-isomorphic (in a suitably defined sense) to a direct sum $Y_1 \oplus_Z Y_2$ with respect to a 1-unconditional basis.

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- After that, we can (almost) follow the plan outlined at the beginning, adding some necessary technical details.

Looking for unconditional sub-sums

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- **Definition:** Let $Y_1 \oplus Y_2$ be a direct sum in which the subspaces Y_1 and Y_2 are Euclidean, and let $\varepsilon \in [0,1)$. The sum $Y_1 \oplus Y_2$ is endowed with a norm whose restrictions to Y_1 and Y_2 are the Euclidean norms. We say $Y_1 \oplus Y_2$ is ε-*invariant* if for any orthogonal operator O_1 on Y_1 and any orthogonal operator O_2 on Y_2, the inequality

$$(1-\varepsilon)\|y_1+y_2\| \le \|O_1y_1+O_2y_2\| \le (1+\varepsilon)\|y_1+y_2\| \quad (4)$$

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- **Lemma 1:** If the sum $Y_1 \oplus Y_2$ is 0-invariant, it is a sum with respect to a 1-unconditional basis in a 2-dimensional space.

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- **Lemma 1:** If the sum $Y_1 \oplus Y_2$ is 0-invariant, it is a sum with respect to a 1-unconditional basis in a 2-dimensional space.
- **Lemma 2:** If the sum $Y_1 \oplus Y_2$ is ε-invariant, it is $(1+\varepsilon)$-isomorphic to 0-invariant.

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- **Proposition:** Consider a Banach space X which has a 1-codimensional subspace Y_0 which is isometric to ℓ_2^N, and its complement is a linear span of x_1. Let $\varepsilon > 0$, there exists a subspace Y_1 with controlled dimension such that the norm of $\alpha x_1 + y$ $(y \in Y_1)$ is ε-invariant to with respect to orthogonal operators on Y_1. This means

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- The proof is based on the Larman-Mani (1975) theorem on almost-spherical sections of non-symmetric convex bodies with center in a given point, with better estimates and a different proof by Gordon (1988).

- Thank you!