Separable faces and renormings of non-separable Banach spaces

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Contents

The coauthors

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LUR norms

Definition

If $(E, \|\cdot\|)$ is a normed space, the norm $\|\cdot\|$ is said to be **locally uniformly rotund** (**LUR**, for short) if

$$\left[\lim_n \left(2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2\right) = 0\right] \Rightarrow \lim_n \|x - x_n\| = 0 \tag{1}$$

for any sequence $\{x_n\}_{n=1}^{\infty}$ and any x in E.

An equivalent, more geometrical, definition of the LUR property of the norm reads: If $\{x, x_1, x_2, \ldots\} \subset S_E$ and $\|x + x_n\| \to 2$, then $\|x - x_n\| \to 0$.

$$\boxed{Q_{\|\cdot\|}(x, y) := 2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2.} \tag{2}$$

A main result

Theorem

A Banach space E with a norming subspace $F \subset E^*$ has an equivalent $\sigma(E, F)$-lower semicontinuous **LUR** *norm if, and only if,there is a sequence $\{A_n : \ n = 1, 2, \cdots\}$ of subsets of E such that, given any $x \in E$ and $\epsilon > 0$, there is a $\sigma(E, F)$-open half-space H and $p \in \mathbb{N}$ such that $x \in H \cap A_p$ and the slice $H \cap A_p$ can be covered with countable many sets of diameter less than ϵ.*

Therefore, this renorming can be achieved as soon as

$$x \in H \cap A_p \subset S_H^p + B(0, \epsilon),$$

where S_H^p is some separable subset of E, in particular any one with finite Hausdorff measure should be good here.

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Solution to open questions

- This result completely solves four problems asked by Moltó, Troyanski, Valdivia and myself. It generalizes Troyanski's fundamental results (see Chapter IV in Deville, Godefroy, Zizler), and Raja's theorems in LUR renormings, as well as García, Oncina, Troyanski and myself where finite covers were considered too .
- E has an equivalent $\sigma(E, F)$-lower semicontinuous **LUR** norm if, and only if, there is another one with separable denting faces. This add an answer to Lindenstrauss' question for LUR norms
- Banach spaces $C(K)$*, where* K *is a Rosenthal compact space* $K \subset \mathbb{R}^{\Gamma}$ *(i.e., a compact space of Baire one functions on a Polish space* Γ*,) with at most countably many discontinuity points for every* $s \in K$*,* This solves questions of Haydon, Moltó and myself.

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Definition

ordan content and Hausdorff Measures We say that a subset S of a metric space (X, d) has δ**-finite Jordan content** if there is a finite sequence of sets

$$\{A_n : n = 1, 2, \cdots, N\}$$

with

$$S \subset \bigcup_{n=1}^{N} A_n,$$

and

$$\text{diam}(A_n) \leq \delta, n = 1, 2, \cdots, N.$$

Jordan content and Hausdorff Measures

Definition

We say that a subset S of a metric space (X, d) has **δ-finite α-dimensional Hausdorff measure** if there is sequence of sets $\{A_n\}_{n=1}^{\infty}$ such that

$$S \subset \bigcup_{n=1}^{\infty} A_n,$$

$$\sum_{n=1}^{\infty} \mathrm{diam}(A_n)^{\alpha} < +\infty$$

and

$$\mathrm{diam}(A_n) \leq \delta, \ \ n = 1, 2, \ldots$$

Jordan content and Hausdorff Measures

Definition

More generally, write $\mathcal{H}_0$ for the class of functions $h : [0, +\infty) \longrightarrow [0, +\infty]$ monotonic increasing for $t \geq 0$, positive for $t > 0$, with $h(0) = 0$, and continuous on the right for all $t \geq 0$. Fixed $h \in \mathcal{H}_0$ and $\epsilon > 0$ we set

$$\mu_\delta^h(S) := \inf \left\{ \sum_{i=1}^{\infty} h\big(\text{diam}(A_i)\big) : \; S \subset \bigcup_{i=1}^{\infty} A_i, \; \text{diam}(A_i) \leq \delta, \; i \in \mathbb{N} \right\}$$

and

$$\mu^h(S) := \sup\{\mu_\delta^h(S) : \; \delta < \epsilon\} = \lim_{\delta \longrightarrow 0} \mu_\delta^h(S).$$

Then μ^h is a metric measure on (X, d) where all Borel sets, even Souslin sets, are μ^h-measurable. Let us denote by μ^α the measure associated to the function $h(t) = t^\alpha$. It is known as the **α-dimensional Hausdorff measure**.

Theorem (Open Localization Theorem)

Let A be a bounded subset in E and $\mathcal{C} = \{\Theta_i : i \in I\}$ be $\sigma(E, F)$-closed convex subsets of E.
Then, there is an equivalent $\sigma(E, F)$-lower semicontinuous norm $\|\cdot\|_{\mathcal{C},A}$ such that:
If $x \in A \setminus \Theta_{i_0}$ for some $i_0 \in I$, and $\{x_n\}_{n=1}^{\infty}$ is a sequence in E such that $\lim_n Q_{\|\cdot\|_{\mathcal{C},A}}(x_n, x) = 0$, then there is a sequence $\{i_n\}_{n=1}^{\infty}$ in I such that:
There is $n_0 \in \mathbb{N}$ such that $x \in A \setminus \Theta_{i_n}$ for each $n \geq n_0$.
Moreover, if for some $n \geq n_0$ we have $x_n \in A$, then $x_n \in A \setminus \Theta_{i_n}$.

Theorem (Open localization plus aproximation theorem)

Let A be a bounded subset in E and $\mathcal{C} := \{\Theta_i : i \in I\}$ be a family of convex and $\sigma(E,F)$-closed subsets of E.
Then there is an equivalent $\sigma(E,F)$-lower semicontinuous norm $\|\cdot\|_{\mathcal{C},A}$ on E such that given $x \in A \setminus \Theta$ for some $\Theta \in \mathcal{C}$ and a sequence $\{x_n\}_{n=1}^{\infty}$ in E with $\lim_n Q_{\|\cdot\|_{\mathcal{C},A}}(x_n, x) = 0$, then there is a sequence $\{i_n\}_{n=1}^{\infty}$ in I verifying the two following properties:

- (i) *There is $n_0 \in \mathbb{N}$ with $x \in A \setminus \Theta_{i_n}$ for each $n \geq n_0$. Moreover, if $x_n \in A$ for some $n \geq n_0$, then $x_n \in A \setminus \Theta_{i_n}$.*
- (ii) *Additionally we will still have the following approximation: For every $\delta > 0$ there is some $n_\delta \in \mathbb{N}$ such that*

$$x, x_n \in \overline{\mathrm{co}(A \setminus \Theta_{i_n}) + \delta B_E}^{\sigma(E,F)} \text{ for all } n \geq n_\delta. \tag{3}$$

Theorem (Δ-Convex Networking)

The following are equivalent:
(i) *E admits a $\sigma(E, F)$-lower semicontinuous equivalent LUR norm.*
(ii) *If $\{A_n\}_{n=1}^{\infty}$ denotes the sequence of balls centered at 0 and having rational radius, and $\mathcal{H}$ denotes the family of all open half-spaces defined by elements in F, then the family of sets $\{A_n \cap H : \ H \in \mathcal{H}, \ n \in\}$ is a network for the norm topology in E.*
(iii) *There is a sequence $\{A_n\}_{n=1}^{\infty}$ of $\sigma(E, F)$-closed convex subsets of E such that the family of sets*

$$\{A_n \setminus \Theta : \ \Theta \in \mathcal{C}, n \in \mathbb{N}\}$$

is a network for the norm topology in E.
(iv) *There is a sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of E such that the family of sets $\{A_n \setminus \Theta : \ \Theta \in \mathcal{C}, n \in \mathbb{N}\}$ is a network for the norm topology in E.*

Lindenstrauss' Question

Question: Characterize those Banach spaces which have an equivalent strictly convex norm.
It is easily verified that every separable Banach space has an strictly convex norm. The same is true for a general WCG space. On the other hand, it was shown by Day that there exist Banach spaces which do not have an equivalent strictly convex norm.
Some conjectures concerning a possible answer to the question were shown to be false by Dashiell and Lindenstrauss.
This results shows that even for $C(K)$ spaces it seems to be a delicate and presumably difficult question to decide under which condition there exists an equivalent strictly convex norm.

Definition

We say that a topological space (X, τ) **is a $T_0(*)$-space** or that the topology τ **is $T_0(*)$** if there is a system $\{\mathcal{W}_n : n \in \mathbb{N}\}$, where each $\mathcal{W}_n$ is a family of open sets, such that for $x \neq y$ there is some $p \in \mathbb{N}$ for which either we have $y \notin \mathrm{Star}(x, \mathcal{W}_p) \neq \emptyset$ or $x \notin \mathrm{Star}(y, \mathcal{W}_p) \neq \emptyset$.

For a family $\mathcal{F}$ of subsets of X, let us remind you:

$$\mathrm{Star}(x, \mathcal{F}) := \bigcup \{F : x \in F \in \mathcal{F}\}.$$

Systems $\{\mathcal{W}_n : n \in \mathbb{N}\}$ are said to **$T_0(*)$- separate points of E**. For a system $\{\mathcal{G}_n : n \in \mathbb{N}\}$, where each $\mathcal{G}_n$ consists of functions from E into $\mathbb{R}$, we say that $\{\mathcal{G}_n : n \in \mathbb{N}\}$ **$T_0(*)$- separates points of E** whenever the system $\{\mathcal{O}_n : n\ in \mathbb{N}\}$ $T_0(*)$-separates points of E, where $\mathcal{O}_n := \{O_g : g \in \mathcal{G}_n\}$ for $n \in \mathbb{N}$, and

$$O_g := \{x \in E : g(x) > 0\}. \tag{4}$$

The Solution

Theorem (Strictly Convex Renorming)

Let E be a normed space with a norming subspace $F \subset E^*$. Then E admits an equivalent $\sigma(E, F)$-lower semicontinuous and strictly convex norm if, and only if, there are families $\mathcal{G}_n$, $n \in \mathbb{N}$, of $\sigma(E, F)$-lower semicontinuous quasi-convex functions defined on E such that the system $\{\mathcal{G}_n : n \in \mathbb{N}\}$ $T_0(*)$-separates points of E.

Answer to Lindenstrauss

Theorem

Let E be a normed space with a norming subspace F in E^*. E admits an equivalent $\sigma(E, F)$-lower semicontinuous rotund norm if, and only if, it has another one with $\sigma(E, F)$-closed and norm separable faces.

Theorem (Bing-Nagata-Smirnov meeting weak topologies)

Let E be a normed space and F a norming subspace of E^*. Then E admits an equivalent $\sigma(E,F)$-lower semicontinuous LUR norm if, and only if, it has a σ-discrete and $\sigma(E,F)$-slicely isolated basis of the norm topology.
In a dual space E^* we have a dual LUR norm if, and only if, the norm topology admits a σ-discrete and w^*-isolated basis.

Theorem (Slicing with finite Hausdorff measure sets)

Let E be a normed space and F a norming subspace of E^*. The space E admits an equivalent $\sigma(E,F)$-lower semicontinuous LUR norm if, and only if, for every $\epsilon > 0$ there is a sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of E such that, for every $x \in E$ and $\epsilon > 0$, there is a $\sigma(E,F)$-open half-space H and an integer n such that

$$x \in A_n \cap H \subset S_n^H + B(0,\epsilon),$$

where S_n^H is a set with finite Hausdorff measure (i.e. $\mu^h(S_n^H) < +\infty$ for some $h \in \mathcal{H}_0$).

Corollary

The normed space E admits an equivalent $\sigma(E,F)$-lower semicontinuous LUR norm if, and only if, for every $\epsilon > 0$ there ais a sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of E such that, for every $x \in E$ and $\epsilon > 0$, there is a $\sigma(E,F)$-open half-space H and an integer n such that $x \in A_n \cap H \subset S_n^H + B(0,\epsilon)$, where S_n^H is a set with zero finite-dimensional Hausdorff measure.

Corollary

E has an equivalent $\sigma(E,F)$-lower semicontinuous LUR norm if, and only if, there is a sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of E such that, given any $x \in E$ and $\epsilon > 0$, there is a $\sigma(E,F)$-open half-space H, $n \in \mathbb{N}$, a sequence $\{x_m\}_{m=1}^{\infty}$ of points in E and an element $(r_m) \in c_0$ such that $x \in H \cap A_n$ and

$$H \cap A_n \subset \bigcap_{m=1}^{\infty} \bigcup_{p=m}^{\infty} B(x_p, r_p + \epsilon)$$

Theorem (The main result)

A Banach space E, with a norming subspace $F \subset E^*$, has an equivalent $\sigma(E, F)$-lower semicontinuous LUR norm if, and only if:
There is a sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of E such that, given any $x \in E$ and $\epsilon > 0$, there is a $\sigma(E, F)$-open half-space H and $n \in \mathbb{N}$ with

$$x \in H \cap A_n \subset S_n^H + B(0, \epsilon)$$

where S_n^H is a separable subset of E .

Corollary

A Banach space E with a norming subspace $F \subset E^*$, has an equivalent $\sigma(E, F)$-lower semicontinuous LUR norm if, and only if, it has anoher one with separable denting faces of its closed unit ball.

Lemma

Let A be a bounded subset in E and $\mathcal{C} = \{\Theta_i : \ i \in I\}$ be a family of convex and $\sigma(E, F)$-closed subsets of E such that, for $i \in I$,

$$A \setminus \Theta_i \subset S_i + B(0, \epsilon),$$

where each S_i has δ-finite Jordan content.
Then there is an equivalent $\sigma(E, F)$-lower semicontinuous norm $\| \cdot \|_{\mathcal{C},A}$ on E with the following property:
For $i \in I$, $x_0 \in A \setminus \Theta_i$, and a sequence $\{x_n\}_{n=1}^{\infty}$ in E such that

$$Q_{\| \cdot \|_{\mathcal{C},A}}(x_0, x_n) \to 0,$$

there exists $n_0 \in \mathbb{N}$ such that

$$\|x_0 - x_n\| \leq 3(\delta + \epsilon), \ \text{for } n \geq n_0.$$

Theorem

Let E be a normed space with F a norming subspace ot its dual. Then E admits an equivalent $\sigma(E, F)$-lower semicontinuous and **LUR** *norm if, and only if, for every $\epsilon > 0$ we can write*

$$E = \bigcup_{n=1}^{\infty} E_{n,\epsilon}$$

in such a way that for every $x \in E_{n,\epsilon}$, there exists a $\sigma(E, F)$-open half-space H containing x with $\alpha(H \cap E_{n,\epsilon}) < \epsilon$, where α denotes the Kuratowski index of non-compactness.

The Proofs

Lemma

Let A be a bounded subset in E and $\mathcal{C} = \{\Theta_i : \ i \in I\}$ be a family of convex and $\sigma(E, F)$-closed subsets of E such that, for every $i \in I$,

$$\emptyset \neq A \setminus \Theta_i \subset S_i + B(0, \epsilon),$$

where S_i a perfect Polish subset of the Banach space E. Then there is an equivalent $\sigma(E, F)$-lower semicontinuous norm $\| \cdot \|_{\mathcal{C},A}$ with the following property:
For every $x_0 \in A \setminus \Theta_{i_0}$ for some $i_0 \in I$ and every sequence $\{x_n\}_{n=1}^{\infty}$ in E such that

$$Q_{\|\cdot\|_{\mathcal{C},A}}(x_0, x_n) \to 0, \tag{5}$$

we should have

$$\|x_0 - x_n\| \leq 5\epsilon$$

for n big enough.

Theorem

Let E be a Banach space, and let $F \subset E^*$ be a norming subspace of E^*. Let us assume that for every $\epsilon > 0$ and every $x \in S_E$ there is a $\sigma(E, F)$-open half-space H with $x \in H$ and $B_E \cap H$ is covered with countably many sets of diameter less or equal to ϵ. Then E admits an equivalent $\sigma(E, F)$-lower semicontinuous and locally uniformly rotund norm.

Theorem (Gluing separable pieces)

Let E be a normed space and F a norming subspace of E^*. The space E admits an equivalent $\sigma(E,F)$-lower semicontinuous LUR norm if, and only if, for every $\epsilon > 0$ there is a family

$$\mathcal{F}_\epsilon := \{F_i^\epsilon : \ i \in I_\epsilon\}$$

of ϵ-separable subsets of B_E with $S_E \subset \bigcup\{F_i^\epsilon : \ i \in I_\epsilon\}$, and such that $\mathcal{F}_\epsilon$ is σ-slicely isolated for $\sigma(E,F)$.
(i.e., the index set I_ϵ can be decomposed as $I_\epsilon = \bigcup_{n=1}^\infty I_n^\epsilon$ in such a way that, for every $n \in \mathbb{N}$, $i_0 \in I_n^\epsilon$, and $x_0 \in F_{i_0}^\epsilon$, there is a $\sigma(E,F)$-open half-space H such that $x_0 \in H$ although $H \cap F_j^\epsilon = \emptyset$ for every $j \in I_n^\epsilon$, $j \neq i_0$).

Theorem (Separable + σ-slicely isolated faces)

Let E be a normed space. Let us assume we have a family $\mathcal{F}$ of separable faces of its unit ball which is σ-slicely isolated. Then the normed space E admits an equivalent locally uniformly rotund norm at every point of $\bigcup \mathcal{F}$.

Theorem

Let E be a Banach space and F a norming subspace in E^*. Let $\mathcal{F}$ be a family of separable, $\sigma(E, F)$-compact faces such that $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ where every subfamily $\mathcal{F}_n$ is a weakly-isolated subfamily. Then the Banach space E admits an equivalent locally uniformly rotund and $\sigma(E, F)$-lower semicontinuous norm on $\bigcup \mathcal{F}$.

Rigidity through δ-disjointness

Definition

Two disjoint faces

$$A := \{x \in B_E : f^*(x) = 1\} \text{ and } B := \{x \in B_E : g^*(x) = 1\}$$

for $f^*, g^* \in S_{E^*}$ in a normed space E are said to be δ-disjoint, for some $\delta > 0$, if the following happens:

- $A - \delta := \{x \in B_E : E : f^*(x) \geq 1 - \delta\}$ doest not meet B and
- $B - \delta := \{x \in B_E : E : g^*(x) \geq 1 - \delta\}$ doest not meet A.

Press down LUR renorming

Theorem

Let E be a Banach space and F a norming subspace in E^*. Let us fix a a non void familly of δ- disjoint $\sigma(E, F)$-closed and separable faces $\mathcal{J}$ of the uni ball B_E for some fixed $\delta > 0$. Then E admits an equivalent $\sigma(E, F)$-lower semicontinuous norm which is going to be locally uniformly rotund at every point of

$$\bigcup\{G : G \in \mathcal{J}\}.$$

Even more, if there is a countable family of sets of faces $\{\mathcal{J}_n : n = 1, 2, \cdots\}$ where every one of the sets $\mathcal{J}_n$ is formed by δ_n- disjoint $\sigma(E, F)$-closed and separable faces for some $\delta_n > 0$, then the norm can be constructed to be LUR on

$$\bigcup\{G : G \in \mathcal{J}_n : n = 1, 2, \cdots\}$$

too.

Press down LUR renorming

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$$\bigcup\{G : G \in \mathcal{J}_n : n = 1, 2, \cdots\}$$

too.

LUR renorming on descriptive Banach spaces

Theorem

Let E be a Banach space and F a norming subspace in E^* such that the identity map on E is $\sigma(E,F)$ to norm σ-continuous; i.e .E is $\sigma(E,F)$-descriptive.
Let $\mathcal{F}$ be a family of separable, $\sigma(E,F)$-compact faces such that $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ where every subfamily $\mathcal{F}_n$ is a norm relatively discrete subfamily. Then the Banach space E admits an equivalent locally uniformly rotund and $\sigma(E,F)$-lower semicontinuous norm on $\bigcup \mathcal{F}$.

LUR renorming on descriptive Banach spaces

Corollary

Let E be a Banach space and F a norming subspace in E^* such that the identity map on E is $\sigma(E, F)$ to norm σ-continuous; i.e .E is $\sigma(E, F)$-descriptive.
If E admits an equivalent $\sigma(E, F)$-strictly convex norm it also has another one being LUR on a norm dense subset too.

Answer to Lindenstrauss

Theorem

Let E be a normed space with a norming subspace F in E^*. Let B_E be the $\sigma(E,F)$-closed, convex unit ball of E and $\mathcal{H} = \{H_i : \ i \in I\}$ be a family of $\sigma(E,F)$-closed afine hyperplanes of E such that

$$S_E \cap H_i$$

is non void but separable for every $i \in I$. Then there is an equivalent $\sigma(E,F)$-lower semicontinuous norm $\|\cdot\|_{\mathcal{H},A}$ with the following property: For every $x \in A \cap H_{i_0}$ for some $i_0 \in I$ and every $y \in E$ such that

$$\left(\frac{1}{2}\|x\|^2_{\mathcal{H},A} + \frac{1}{2}\|y\|^2_{\mathcal{H},A} - \left\|\frac{x+y}{2}\right\|^2_{\mathcal{H},A}\right) = 0, \tag{6}$$

we should have $x = y$.

Answer to R. Smith

Theorem

Let E^* be a dual Banach space. Then E^* admits a dual rotund equivalent norm if, and only if, its unit ball B_{E^*} has $T_0(*)$ for the w^* topology

Corollary

Dual Banach spaces wich are w^*-homeomorphic preserve the property of being dual strictly convex renormable.

Historical context of our research

LUR-renorm. $\rightarrow$ Kadec-renorm.$\rightarrow$ Descriptive space$\rightarrow$ weakly Cech-analytic $\rightarrow$ σ-fragmentable

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