On the complemented subspace problem

based on joint work with D. de Hevia, A. Salguero-Alarcón and P. Tradacete

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Linear and non-linear analysis in Banach Spaces

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CSP for $C(K)$-spaces. Are complemented subspaces in $C(K)$-spaces isomorphic to $C(K)$-spaces?

One of the most important problems in the theory of Banach lattices, which is still open, is whether any complemented subspace of a Banach lattice must be linearly isomorphic to a Banach lattice.

P. Casazza, N. Kalton and L. Tzafriri (1987)

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CSP for Banach lattices. Are complemented subspaces in Banach lattices isomorphic to Banach lattices?

Theorem (G. Plebanek, A. Salguero-Alarcón, 2023)

There exist compact spaces K and L such that $\mathcal{C}(L) = \mathcal{C}(K) \oplus X$ but X is not isomorphic to a space of continuous functions.

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In fact, the space X is not isomorphic to a Banach lattice.

Definition. A *Banach lattice* is a Banach space $(X, \|\cdot\|)$ equipped with a lattice order $\leq$ which satisfies

1. If $x \leq y$, then $x + z \leq y + z$ and $ax \leq ay$ for any $a \in \mathbb{R}^+$;
2. If $|x| \leq |y|$, then $\|x\| \leq \|y\|$.

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Compact spaces induced by almost disjoint families

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If $\mathcal{F}$ is an almost disjoint family, the compact space induced by $\mathcal{F}$ is the compact space $K_{\mathcal{F}} = \mathbb{N} \cup \{p_A : A \in \mathcal{F}\} \cup \{\infty\}$ where:

- points in $\mathbb{N}$ are isolated;
- given $A \in \mathcal{F}$, a basic neighbourhood of p_A is of the form $\{p_A\} \cup (A \setminus F)$, where F is finite;
- $K_{\mathcal{F}}$ is the one-point compactification of the locally compact space $\mathbb{N} \cup \{p_A : A \in \mathcal{F}\}$.

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Desired Property (DP)

For every norming sequence $(e_n^*)_{n\in\mathbb{N}}$ in X^* there exists an element $x \in X$ such that no element $y \in X$ satisfies

$$e_n^*(y) = |e_n^*(x)| \text{ for every } n \in \mathbb{N}.$$

Open Problem

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Thank you for your attention.