Slicely countably determined points in Banach spaces

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July 18th, 2023
XXII Luís Santaló School
Linear and non-linear analysis in Banach spaces
Santander, Spain

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Outline of the talk

This work was supported by the Estonian Research Council grant (PSG487).

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Notation

- X real or complex Banach space, X^* dual space
- S_X unit sphere, B_X closed unit ball
- $\mathrm{conv}(\cdot)$ convex hull, $\overline{\mathrm{conv}}(\cdot)$ closed convex hull

A slice of A (bounded convex $\subset X$) is a (nonempty) subset of the form

$$S(A, x^*, \alpha) := \{x \in A : \mathrm{Re}\, x^*(x) > \sup \mathrm{Re}\, x^*(A) - \alpha\} \quad (x^* \in X^*, \alpha > 0)$$

A convex combination of slices of A is the set

$$\sum_{i=1}^{n} \lambda_i S(A, x_i^*, \alpha_i),$$

where $\lambda_1, \ldots, \lambda_n \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$.

SCD sets and spaces

Let $A \subset X$ be bounded and convex.

Definition (A. Avilés, V. Kadets, M. Martín, J. Merí, V. Shepelska (2010))

A sequence $\{V_n \colon n \in \mathbb{N}\}$ of subsets of A is determining for A, if one of the following equivalent conditions hold:

- if $B \subset A$ satisfies $B \cap V_n \neq \emptyset$ for every n, then $A \subset \overline{\mathrm{conv}}(B)$;
- if $x_n \in V_n$ for every n, then $A \subset \overline{\mathrm{conv}}(\{x_n \colon n \in \mathbb{N}\})$;
- if for every slice S of A, there is a V_m such that $V_m \subset S$.

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- if for every slice S of A, there is a V_m such that $V_m \subset S$.

Definition (AKMMS (2010))

The set A is said to be slicely countably determined (an SCD set in short), if there exists a determining sequence of slices of A.

Properties and positive examples of SCD sets

Proposition (AKMMS (2010))

- *A is SCD iff $\overline{A}$ is SCD*
- *If A is an SCD set, then A is separable*

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Example (AKMMS (2010))

If A is separable and $A = \overline{\mathrm{conv}}(\mathrm{dent}(A))$, then A is SCD. In particular, if X has RNP, then every closed and bounded subset is SCD.

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Corollary (AKMMS (2010))

If X is separable and LUR, then B_X is SCD. Hence, every separable space can be renormed such that $B_{(X,|\cdot|)}$ is SCD.

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If X is separable and LUR, then B_X is SCD. Hence, every separable space can be renormed such that $B_{(X,|\cdot|)}$ is SCD.

Example (AKMMS (2010))

If X^* is separable, then every A is SCD.

Negative examples

Example (AKMMS (2010))

If X has the Daugavet property, then B_X is not SCD.

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A subset of an SCD set need not to be an SCD set.

Proof.

Consider $X := C[0,1]$. Since X is separable, it admits an equivalent LUR renorming. Hence $B_{(X,|\cdot|)}$ is SCD, but there is a $\alpha \in \mathbb{R}$ such that

$$\alpha B_{(X,\|\cdot\|)} \subset B_{(X,|\cdot|)}$$

and $B_{(X,\|\cdot\|)}$ is not SCD. □

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Example (V. Kadets, M. Martin, J. Meri, D. Werner (2013))

If X has a 1-unconditional basis, then B_X is SCD.

Problem

If X has a 1-unconditional basis, then X is an SCD space?

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Determining sequence

Let $A \subset X$ be bounded and convex.

Definition

We say that a countable sequence $\{V_n\colon n \in \mathbb{N}\}$ of subsets of A is determining for point $a \in A$ if $a \in \overline{\mathrm{conv}}(B)$ for every $B \subset A$ intersecting all the sets V_n.

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Proposition

For a sequence $\{V_n \colon n \in \mathbb{N}\}$ of subsets of A, the following conditions are equivalent:

(i) $\{V_n \colon n \in \mathbb{N}\}$ *is determining for a;*

(ii) *for every slice S of A with $a \in S$, there is $m \in \mathbb{N}$ such that $V_m \subset S$;*

(iii) *if $x_n \in V_n$ for every $n \in \mathbb{N}$, then $a \in \overline{\mathrm{conv}}(\{x_n \colon n \in \mathbb{N}\})$.*

SCD points

Definition

A point $a \in A$ is called a slicely countably determined point of A (an SCD point of A in short), if there exists a determining sequence of slices of A for the point a.

We denote the set of all SCD points of A by $\mathrm{SCD}(A)$.

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Lemma

The following statements hold:

1. *If A is an SCD set, then every $a \in A$ is an SCD point.*
2. *If every $a \in A$ is an SCD point and A is separable, then A is an SCD set.*

First examples of SCD points

Proposition

The following conditions are equivalent:

(i) $a \in \mathrm{SCD}(A)$;

(ii) *there exists a sequence of relatively weakly open sets* $\{W_n \colon n \in \mathbb{N}\} \subset A$, *which is determining for* a;

(iii) *there exists a sequence of convex combinations of slices* $\{C_n \colon n \in \mathbb{N}\} \subset A$, *which is determining for* a.

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A point a of a closed bounded and convex set A is called a strongly regular point of A if for every $\varepsilon > 0$ there exists a convex combination C of slices of A such that $a \in \overline{C}$ and $\mathrm{diam}(C) < \varepsilon$ (**H. Rosenthal** (1988)).

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Lemma

If $a \in A$ *is a strongly regular point (in particular, denting point), then* a *is an SCD point.*

SCD points vs Daugavet points

Recall that (**T. A. Abrahamsen, R. Haller, V. Lima, K. Pirk** (2020))

- $x \in S_X$ is a Daugavet point if for every slice S of B_X and for every $\varepsilon > 0$ there is a $y \in S$ with $\|x - y\| \geq 2 - \varepsilon$.
- X has the Daugavet property if every $x \in S_X$ is a Daugavet point.

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Example

The constant function $\mathbf{1}$ is both a Daugavet point and an SCD point in B_c.

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There exists a Banach space (actually a Lipschitz-free space) with the RNP and a Daugavet point.

$x_0 \in \mathrm{SCD}(A)$ if $\exists \{S_n\} \subset A$ such that $\forall x_0 \in S \subset A$ there is $S_m \subset S$

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Example

Assume that X has the RNP and let $x_0 \in S_X$ be a Daugavet point. Then for every $\varepsilon > 0$ there exists a sequence of slices $\{S_n \colon n \in \mathbb{N}\} \subset B_X$ determining for x_0 such that $d(x_0, S_n) > 2 - \varepsilon$ for every $n \in \mathbb{N}$.

Properties of $\mathrm{SCD}(A)$

Lemma

Let $A \subset X$ be bounded and convex. Then

- $\mathrm{SCD}(A)$ *is convex and norm closed.*
- *if A is balanced, then so is* $\mathrm{SCD}(A)$.

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Corollary

$\mathrm{SCD}(B_X) \neq \emptyset$ *if and only if* $0 \in \mathrm{SCD}(B_X)$.

When is $\mathrm{SCD}(B_X) = \emptyset$?

A Banach space X is said to have property $(\approx\approx)$ if for all sequences of slices $\{S_n \colon n \in \mathbb{N}\}$ of B_X, there are $\gamma \in (0,1]$, $x_n \in S_n$, and $x^* \in S_{X^*}$ such that $\operatorname{Re} x^*(x_n) \geq \gamma$ for every $n \in \mathbb{N}$.

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Remark

If $\gamma = 1$, then $(\approx\approx) = 1\text{-ASD2P}_\omega$ (**S. Ciaci, J. L., A. Lissitsin** (2022))

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- $X \oplus_p Y$ has $(\approx\!\approx)$ whenever X and Y have $(\approx\!\approx)$.

Theorem

If a Banach space X has property $(\approx\!\approx)$, then $\mathrm{SCD}(B_X) = \emptyset$.

When is $\mathrm{SCD}(B_X) = \emptyset$?

Definition (A. Guirao, A. Lissitsin, V. Montesinos (2019))

A Banach space X is said to fail (-1)-BCP if for any separable subspace $Y \subset X$ there exists $x \in S_X$ such that equality

$$\|y + \lambda x\| = \|y\| + |\lambda| \tag{2.1}$$

holds for every $y \in Y$ and $\lambda \in \mathbb{R}$.

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Example (GLM (2019) + S. Ciaci, J. L., A. Lissitsin (2023))

Examples of Banach spaces failing the (-1)-BCP include $\ell_1(I)$, where I is an uncountable set, the space ℓ_∞ / c_0, and X^* whenever X has the Daugavet property.

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The following chain of implications holds:

$$X \text{ is Daugavet} \Rightarrow X^* \text{ fails (-1)-BCP} \Rightarrow X \text{ has } (\approx\approx) \Rightarrow \mathrm{SCD}(B_X) = \emptyset$$

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SCD points in L_1-preduals

Recall that given a measured space (S, Σ, μ) a Banach space X is said to be an L_1-predual, if $X^* = L_1(S, \Sigma, \mu)$. It is known that

$$L_1(S, \Sigma, \mu) = L_1(S, \Sigma, \nu) \oplus_1 \ell_1(I), \tag{3.1}$$

where ν is a non-atomic measure and I is some index set.

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where ν is a non-atomic measure and I is some index set.
For $f, g \in \operatorname{ext}(B_{X^*})$, we say $f \sim g$ if and only if $f = \lambda g$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. We denote the quotient set by $\operatorname{ext}(B_{X^*})/\sim$.

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Theorem

Let X be an L_1-predual. Then the following statements hold:

(a) *If $\operatorname{ext}(B_{X^*})/\sim$ is at most countable, then X is an SCD space. In particular, $\mathrm{SCD}(B_X) = B_X$;*

(b) *If $\operatorname{ext}(B_{X^*})/\sim$ is uncountable, then $\mathrm{SCD}(B_X) = \emptyset$.*

SCD points in L_1-preduals

Proof.

(*a*). If $\operatorname{ext}(B_{X^*})/\sim$ is at most countable, then it is known that X^* is separable. Thus X^* has the RNP, and hence X is an SCD space (Ex. 3.2, AKMMS), in particular, $\mathrm{SCD}(B_X) = B_X$.

Proof.

(a). If $\operatorname{ext}(B_{X^*})/\sim$ is at most countable, then it is known that X^* is separable. Thus X^* has the RNP, and hence X is an SCD space (Ex. 3.2, AKMMS), in particular, $\mathrm{SCD}(B_X) = B_X$.
(b). Assume that $\operatorname{ext}(B_{X^*})/\sim$ is uncountable and consider the decomposition (3.1). It is known that $\operatorname{ext}(B_{L_1(S,\Sigma,\nu)}) = \emptyset$ whenever ν is a non-atomic measure. Thus

$$\operatorname{ext}(B_{X^*}) \subset \{0\} \times \operatorname{ext}(B_{\ell_1(I)}). \tag{3.2}$$

Moreover, $\operatorname{ext}(B_{\ell_1(I)}) = \{\lambda e_i \colon i \in I, |\lambda| = 1\}$, where $e_i(j) = \delta_{ij}$. Now since the set $\operatorname{ext}(B_{X^*})/\sim$ is uncountable, we deduce that I is uncountable (by (3.2)). It is known that $\ell_1(I)$ then fails (-1)-BCP and then by (CLL (2023)) the absolute sum $L_1(S,\Sigma,\nu) \oplus_1 \ell_1(I)$ also fails (-1)-BCP. Thus, we obtain that $\mathrm{SCD}(B_X) = \emptyset$. □

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SCD points in ℓ_p-sums

Let X and Y be Banach spaces

Theorem

Then $(a, b) \in \mathrm{SCD}(B_{X \oplus_\infty Y})$ *if and only if* $a \in \mathrm{SCD}(B_X)$ *and* $b \in \mathrm{SCD}(B_Y)$.

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Proposition

Let $1 \leq p < \infty$.

(a) *If* $a \in \mathrm{SCD}(B_X)$, *then* $(a,0) \in \mathrm{SCD}(B_{X\oplus_p Y})$.

(b) *If* $a \in S_X$ *and* $(a,0) \in \mathrm{SCD}(B_{X\oplus_p Y})$, *then* $a \in \mathrm{SCD}(B_X)$.

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(b) *If* $a \in S_X$ *and* $(a,0) \in \mathrm{SCD}(B_{X\oplus_p Y})$, *then* $a \in \mathrm{SCD}(B_X)$.

Theorem

Let $(a,b) \in S_{X\oplus_1 Y}$, *where* $a \in X \setminus \{0\}$ *and* $b \in Y \setminus \{0\}$. *Then* $(a,b) \in \mathrm{SCD}(B_{X\oplus_1 Y})$ *if and only if* $\frac{a}{\|a\|} \in \mathrm{SCD}(B_X)$ *and* $\frac{b}{\|b\|} \in \mathrm{SCD}(B_Y)$.

A Banach space where $\mathrm{SCD}(B_X) = \{0\}$

Let (X_n) be Banach spaces. Consider $X := \left(\bigoplus_{n=1}^{\infty} X_n\right)_p$ endowed with the norm

$$\|x\| = \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{1/p}, \quad \text{where } x = (x_n)_{n=1}^{\infty} \text{ and } 1 < p < \infty.$$

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If (X_n) is arbitrary, then $0 \in SCD(B_X)$.

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Theorem

If (X_n) is arbitrary, then $0 \in SCD(B_X)$.

Proposition

Assume that $X := E \oplus_p Y$, where E has the Daugavet property, Y is arbitrary, and $1 < p < \infty$. If $(a, b) \in \mathrm{SCD}(B_X)$, then $a = 0$.

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Proposition

Assume that $X := E \oplus_p Y$, where E has the Daugavet property, Y is arbitrary, and $1 < p < \infty$. If $(a, b) \in \mathrm{SCD}(B_X)$, then $a = 0$.

Theorem

Consider the Banach space $X := \left(\bigoplus_{n=1}^{\infty} E_n\right)_p$, where $1 < p < \infty$ and E_n are spaces with the Daugavet property. Then $\mathrm{SCD}(B_X) = \{0\}$.

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SCD points in projective tensor product

Problem

If X and Y are SCD spaces, then so is $X\hat{\otimes}_\pi Y$?

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Let X and Y be real Banach spaces. If $a \in \mathrm{dent}(B_X)$ and $b \in \mathrm{SCD}(B_Y) \setminus \{0\}$, then $a \otimes b \in \mathrm{SCD}(B_{X \hat{\otimes}_\pi Y})$.

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Corollary

Let X and Y be real Banach spaces such that B_X is dentable and $\mathrm{SCD}(B_Y) = B_Y$. Then $\mathrm{SCD}(B_{X\hat{\otimes}_\pi Y}) = B_{X\hat{\otimes}_\pi Y}$. If B_X is also separable and B_Y is an SCD set, then $B_{X\hat{\otimes}_\pi Y}$ is an SCD set.

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SCD points in $\mathcal{F}(M)$

A point $x_0 \in B_X$ is a strongly exposed point if there is a $x^* \in X^*$ such that $\mathrm{diam}(S(B_X, x^*, \alpha)) \to 0$ whenever $\alpha \to 0$.

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Theorem

Let M be a compact metric space and let $\mu \in S_{\mathcal{F}(M)}$. TFAE:

(i) *μ is an SCD point.*

(ii) $\mu \in \overline{\mathrm{conv}}(str.exp(B_{\mathcal{F}(M)})$.

In particular, $B_{\mathcal{F}(M)}$ is SCD if and only if $B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}(str.exp(B_{\mathcal{F}(M)}))$.

SCD points in $\mathcal{F}(M)$

Recall that M is a proper if every closed bounded set is compact. Given two points $x, y \in M$, we write

$$[x, y] := \{z \in M : d(x, z) + d(y, z) = d(x, y)\}.$$

Moreover, M is rotund if, given $R > 0$, the condition $x, y \in B(0, R)$ implies $[x, y] \subseteq B(0, R)$.

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Applications

Let X be a Banach space and let Y be a subspace of X.

Theorem (V. Kadets, V. Shepelska, G. Sirotkin, D. Werner (2000))

If X has the Daugavet property and Y is an M-ideal in X or $(X/Y)^*$ is separable, then Y has the Daugavet property.

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And one more...

Theorem

If X is separable and X^* fails (-1)-BCP, then X contains ℓ_1.

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