Non linear geometry and asymptotic properties of Banach spaces

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Some general questions in non linear geometry of Banach spaces.

In this course, we will only consider vector spaces over $\mathbb{R}$.

Question 1. What are the linear properties of Banach spaces that are stable under non linear embeddings or equivalences (preserving distances in various ways that have to be made precise) ?

Answers. The topology of a Banach space is determined by its dimension when it is finite (Brouwer 1912), by its density character when the dimension is infinite (Kadets 1960, Toruńczyk 1981).
A surjective isometry between normed spaces is affine (Mazur-Ulam 1932).

Question 2. Can we characterize in purely metric terms the linear properties of Banach spaces that are stable under these non linear embeddings ?

Question 3. Does the metric structure of a Banach space determine its linear structure ?

Question 4. For a given type of embedding, describe the separable Banach spaces that are universal for the class of separable metric spaces.

Embeddings and equivalences.

Let X and Y be two Banach spaces.

• We denote $X \simeq Y$ if there exists a (bi)continuous linear bijection from X onto Y. We say that X and Y are isomorphic.

• We denote $X \underset{\sim}{\subseteq} Y$, if X is isomorphic to a subspace of Y.

Let (M, d) and (N, δ) be two metric spaces, $f : M \to N$ and $t \geq 0$.

$\rho_f(t) = \inf\{\delta(f(x), f(x'), \; d(x, x') \geq t\}$ is the compression modulus of f.
$\omega_f(t) = \sup\{\delta(f(x), f(x'), \; d(x, x') \leq t\}$ is the expansion modulus of f.
ω_f and ρ_f are the "best non decreasing functions" such that

$$\forall x, x' \in M \;\; \rho_f(d(x, x')) \leq \delta(f(x), f(x')) \leq \omega_f(d(x, x')).$$

• f is a Lipschitz embedding if there exist $A, B > 0$ such that $\omega_f(t) \leq At$ and $\rho_f(t) \geq Bt$, for all $t \geq 0$. We denote $M \underset{L}{\hookrightarrow} N$.

• f is coarse Lipschitz if there exist $A, B > 0$ such that $\omega_f(t) \leq At + B$

• f is a coarse Lipschitz embedding if there exist $A, B, C > 0$ such that $\omega_f(t) \leq At + C$ and $\rho_f(t) \geq Bt - C$, for all $t \geq 0$. We denote $M \underset{CL}{\hookrightarrow} N$.

• f is a coarse embedding if $\omega_f < \infty$ and $\lim_\infty \rho = \infty$.
We denote $M \underset{c}{\hookrightarrow} N$.

• f is a uniform embedding if $\rho_f(t) > 0$ for all $t > 0$ and $\lim_{t \to 0} \omega_f(t) = 0$.
We denote $M \underset{u}{\hookrightarrow} N$.

• f is a Lipschitz equivalence if f is bijective and f, f^{-1} are Lipschitz.
We denote $M \underset{L}{\sim} N$.

• f is a uniform homeomorphism if f is bijective and f, f^{-1} are uniformly continuous. We denote $M \underset{UH}{\sim} N$.

• f is a coarse Lipschitz equivalence if f is coarse Lipschitz and there exist $g : N \to M$ coarse Lipschitz and $C > 0$ such that

$$\forall x \in M \;\; d(g \circ f(x), x) \leq C \;\text{ and }\; \forall y \in N \;\; \delta(f \circ g(y), y) \leq C.$$

We denote $M \underset{CL}{\sim} N$.

The Ribe program.

Ribe (1976)

Let X and Y be two Banach spaces such that $X \underset{CL}{\hookrightarrow} Y$.
Then there exists $C \geq 1$ such that for any finite dimensional subspace E of X, there exist a subspace F of Y and an isomorphism $T : E \to F$ with $\|T\| \, \|T^{-1}\| \leq C$.
We say that X is finitely crudely representable into Y.

The "local" properties of Banach spaces are stable under coarse Lipschitz embeddings.

The Ribe program (initiated by Bourgain and Lindenstrauss)

1) Characterize the local properties of Banach spaces in metric terms.
2) Do some well known phenomenons from the local linear theory of Banach spaces extend to the setting of metric spaces ?

Examples : type, cotype, super-reflexivity, UMD...

The Kalton program.

• In the last 25 years the "asymptotic properties" of Banach spaces have provided many linear properties that are invariant under $\underset{L}{\sim}$, $\underset{CL}{\sim}$, $\underset{CL}{\hookrightarrow}$ and even $\underset{c}{\hookrightarrow}$ or $\underset{u}{\hookrightarrow}$. This program was initiated by many powerful results due to Nigel Kalton, sometimes with coauthors.

• Very vaguely speaking, asymptotic properties of Banach spaces are properties related to the structure of finite codimensional spaces, such as properties of weakly null sequences or, more importantly, weakly null trees.

• The main tools used in these results are :

1) Hamming graphs for the study of the stability of asymptotic uniform smoothness under coarse or coarse-Lipschitz embeddings.
2) Hyperbolic trees, coarse-Lipschitz embeddings, property (β) of Rolewicz.
3) The approximate midpoints principle for the study of the stability of asymptotic uniform convexity under coarse-Lipschitz embeddings.
4) Kalton's interlacing graphs in relation with coarse universality and also stability of weak* asymptotic uniform convexity.
5) The Gorelik Principle for the study of the stability of asymptotic uniform smoothness under Lipschitz or coarse Lipschitz equivalences.
6) Lipschitz free spaces.

CHAPTER I.

UNIVERSAL SPACES FOR SEPARABLE METRIC SPACES

Banach-Mazur (1930's)

Every separable metric space isometrically embeds into $C([0,1])$.

Aharoni (1974)

Every separable metric space Lipschitz embeds into c_0. It can be done with distortion 2 and it is optimal.

Problem 1

Let X be a Banach space such that $c_0 \underset{L}{\hookrightarrow} X$. Is c_0 isomorphic to a subspace of X ? Is c_0 a minimal universal space for $\underset{L}{\hookrightarrow}$?

<u>Note.</u> Differentiability theorems imply that such an X cannot have the Radon-Nikodým property (RNP in short). In particular it cannot be a separable dual.

Problem 2

What can we say about a Banach space X such that $c_0 \underset{c}{\hookrightarrow} X$ or $c_0 \underset{CL}{\hookrightarrow} X$? Can it have RNP ? Can it be a separable dual or even reflexive ?

Universal free spaces

Let $N = (\mathbb{Z}^{\mathbb{N}} \cap c_0, \| \ \|_\infty)$. Then $c_0 \underset{CL}{\hookrightarrow} N$ and $c_0 \underset{CL}{\hookrightarrow} \mathcal{F}(N)$.

Kalton (2004)

$\mathcal{F}(N)$ is a Schur space with the RNP (and universal for $\underset{CL}{\hookrightarrow}$).

Let $\alpha \in (0,1)$ and $\omega_\alpha(t) = \max\{t^\alpha, t\}$. If (M,d) is a metric space, so is $(M, \omega_\alpha \circ d)$. We denote $\mathcal{F}_\alpha(M) = \mathcal{F}(M, \omega_\alpha \circ d)$. Then c_0 strongly embeds (that is uniformly and coarse Lipschitz at the same time) into $\mathcal{F}_\alpha(c_0)$.

Kalton (2004) + Aliaga, Gartland, Petitjean, Procházka (2023)

$\mathcal{F}_\alpha(c_0)$ is a Schur space with RNP.

<u>**Consequence :**</u> c_0 does not Lipschitz embed into $\mathcal{F}(N)$, nor into $\mathcal{F}_\alpha(c_0)$.

Problem 3

Do $\mathcal{F}(N)$ and $\mathcal{F}_\alpha(c_0)$ linearly embed into a separable dual ? Does every separable free space with RNP linearly embed into a separable dual ?

Kalton's interlacing metric

Let $\mathbb{M}$ be an infinite subset of $\mathbb{N}$ and $k \in \mathbb{N}$. We denote

$$[\mathbb{M}]^k = \{\overline{n} = (n_1, \dots, n_k),\ n_1 < \cdots < n_k \in \mathbb{M}\}, \quad [\mathbb{M}]^{<\omega} = \bigcup_{k=0}^{\infty} [\mathbb{M}]^k.$$

d_I^k is the graph metric on $[\mathbb{N}]^k$ such that $d_I^k(\overline{n}, \overline{m}) = 1$ iff $\overline{n} \neq \overline{m}$ and $n_1 \le m_1 \le \cdots \le n_k \le m_k$ or $m_1 \le n_1 \le \cdots \le m_k \le n_k$.

Fact

(1) For $\overline{n}, \overline{m} \in [\mathbb{N}]^k$:
$d_I^k(\overline{n}, \overline{m}) = \max\left\{\left|\sharp(\overline{n} \cap S) - \sharp(\overline{m} \cap S)\right|,\ S \text{ segment of } \mathbb{N}\right\}$.
We use this formula to extend d_I to $[\mathbb{N}]^{<\omega}$.

(2) Let (s_n) be the summing basis of c_0 ($s_n = \sum_{i=1}^n e_i$) and define $f(n_1, \dots, n_k) = \sum_{i=1}^k s_{n_i}$. Then

$$\frac{1}{2} d_I(\overline{n}, \overline{m}) \le \|f(\overline{n}) - f(\overline{m})\|_\infty \le d_I(\overline{n}, \overline{m}).$$

Property $\mathcal{Q}_p$

Definition

Let $p \in (1, \infty]$. A Banach space X has property $\mathcal{Q}_p$ if there exists $C \geq 1$ such that for any $k \in \mathbb{N}$ and any $f : ([\mathbb{N}]^k, d_I) \to X$ Lipschitz, there exists $\mathbb{M} \in [\mathbb{N}]^\omega$ so that diam $(f([\mathbb{M}]^k)) \leq Ck^{1/p}\text{Lip}(f)$.
We denote Q_X^p the infimum of those $C \geq 1$.

Proposition

(i) For $p \in (1, \infty)$, $\mathcal{Q}_p$ is stable under coarse Lipschitz embeddings.

(ii) Property $\mathcal{Q}_\infty$ is stable under coarse embeddings. In particular, if a Banach space X has $\mathcal{Q}_\infty$, then c_0 does not coarsely embed into X.

Proof (ii). Assume $Q_Y^\infty = \lambda < \infty$ and $g : X \underset{c}{\hookrightarrow} Y$.
Pick $\mu > 0$ such that $\rho_g(\mu) \geq \lambda\omega_g(1)$.
Let $f : ([\mathbb{N}]^k, d_I) \to X$ 1-Lipschitz. Then $\text{Lip}(g \circ f) \leq \omega_g(1)$.
Thus, there exists $\mathbb{M} \in [\mathbb{N}]^\omega$ so that diam $((g \circ f)([\mathbb{M}]^k)) \leq \lambda\omega_g(1)$.
It follows that $\rho_g(\text{diam }(f([\mathbb{M}]^k))) \leq \rho_g(\mu)$, and diam $(f([\mathbb{M}]^k)) \leq \mu$.
So $Q_X^\infty \leq \mu$. □

Kalton (2007)

Let X be a reflexive Banach space. Then $Q_X^\infty \le 2$.

Proof. Let X a reflexive Banach space and $f : ([\mathbb{N}^k], d_I) \to X$ Lipschitz. Fix $\mathcal{U}$ a non principal ultrafilter on $\mathbb{N}$ and define $\partial f : [\mathbb{N}^{k-1}], d_I) \to X$ by

$$\partial f(n_1, \dots, n_{k-1}) = w - \lim_{n_k \in \mathcal{U}} f(n_1, \dots, n_{k-1}, n_k).$$

Note that $\mathrm{Lip}(\partial f) \le \mathrm{Lip}(f)$ and $\partial^k f \in X$.

Lemma 1

Assume $X = \mathbb{R}$ and let $\varepsilon > 0$. Then there exists $\mathbb{M} \in [\mathbb{N}]^\omega$ such that $|f(\overline{n}) - \partial^k f| \le \varepsilon$, for all $\overline{n} \in [\mathbb{M}]^k$.

Lemma 2

Let $f : ([\mathbb{N}^k], d_I) \to X$ and $g : ([\mathbb{N}^k], d_I) \to X^*$ bounded maps and define $f \otimes g : [\mathbb{N}]^{2k} \to \mathbb{R}$ by
$(f \otimes g)(n_1, \dots, n_{2k}) = \langle f(n_2, \dots, n_{2k}), g(n_1, \dots, n_{2k-1}) \rangle$.
Then $\partial^2(f \otimes g) = \partial f \otimes \partial g$ .... $\partial^{2k}(f \otimes g) = \langle \partial^k f, \partial^k g \rangle$.

Lemma 3

Let $f : ([\mathbb{N}]^k, d_I) \to X$ be a 1-Lipschitz map and $\varepsilon > 0$. Then, there exists an infinite subset $\mathbb{M} \in [\mathbb{N}]^\omega$ such that

$$\forall \overline{n} \in [\mathbb{M}]^k \quad \|f(\overline{n})\| \leq \|\partial^k f\| + \mathrm{Lip}(f) + \varepsilon.$$

Proof. For all $\overline{n} \in [\mathbb{N}]^k$, there exists $g(\overline{n}) \in S_{X^*}$ such that $\langle f(\overline{n}), g(\overline{n})\rangle = \|f(\overline{n})\|$. Then, by Lemma 2,

$$|\partial^{2k}(f \otimes g)| = |\langle \partial^k f, \partial^k g\rangle| \leq \|\partial^k f\|.$$

By Lemma 1, there is an infinite subset $\mathbb{M}_0$ of $\mathbb{N}$ such that

$$\forall \overline{p} \in [\mathbb{M}_0]^{2k} : \ |(f \otimes g)(\overline{p})| \leq \|\partial^k f\| + \varepsilon.$$

Then write $\mathbb{M}_0 = \{n_1 < m_1 < .. < n_i < m_i < ..\}$ and set $\mathbb{M} = \{n_1 < n_2 < .. < n_i < ..\}$. Thus for all $\overline{n} = (n_{i_1}, .., n_{i_k}) \in [\mathbb{M}]^k$,

$$\|f(\overline{n})\| = \langle f(\overline{n}), g(\overline{n})\rangle \leq |\langle f(m_{i_1}, .., m_{i_k}), g(n_{i_1}, .., n_{i_k})\rangle| + \mathrm{Lip}(f)$$
$$\leq \|\partial^k f\| + \varepsilon + \mathrm{Lip}(f).$$

Proof of Theorem. Apply Lemma 3 to $(f - \partial^k f)$.

Kalton (2007)

If c_0 coarsely (or uniformly) embeds into a Banach space X, then there exists $k \in \mathbb{N}$ such that $X^{(n)}$ is non separable.

Definition. A Banach space is said to be *stable* if for all (x_n), (y_n) bounded sequences in X and for $\mathcal{U}$ non principal ultrafilter on $\mathbb{N}$,

$$\lim_{n\in\mathcal{U}}\lim_{m\in\mathcal{U}}\|x_n - y_m\| = \lim_{m\in\mathcal{U}}\lim_{n\in\mathcal{U}}\|x_n - y_m\|.$$

L_p for $p \in [1, \infty)$ is stable (Krivine and Maurey).

Kalton(2007)

(1) If X is stable, then $Q_X^\infty \leq 2$ (easy).
(2) If X is stable, then X strongly embeds into a reflexive Banach space.

Problem 4

(4.1) Does any Banach space with $\mathcal{Q}_\infty$ coarsely embed into a reflexive Banach space ?
(4.2) Does any reflexive Banach space coarsely embed into a stable Banach space ?

More open problems

(1) Does c_0 coarsely or coarse Lipschitz embed into a separable dual? A separable bidual?
(2) Does there exist $n \geq 1$ such that $X^{(n)}$ separable implies that c_0 does not coarsely (or coarse Lipschitz) embed into X?

Funny example

Let $\mathcal{J}$ be the James space. Then
(1) (Kalton 2007). The spaces $\mathcal{J}$ and $\mathcal{J}^*$ fail to have property $\mathcal{Q}_\infty$.
(2) (Petitjean, Procházka, L. (2020). The family of metric spaces $([\mathbb{N}]^k, d_I)_{k\in\mathbb{N}}$ does not equi-coarsely embed into $\mathcal{J}$ or $\mathcal{J}^*$.

Next problem.

To what extent can we equi-Lipschitz embed the family $([\mathbb{N}]^k, d_I)_{k\in\mathbb{N}}$ into a separable dual?

CHAPTER II.

SOME ASYMPTOTIC PROPERTIES OF BANACH SPACES

Moduli of asymptotic uniform smoothness and convexity.

Let $(X, \| \ \|)$ be a Banach space and $t \in [0, \infty)$.

$$\overline{\rho}_X(t) = \sup_{x \in S_X} \inf_{\dim(X/Y)<\infty} \sup_{y \in S_Y} \|x + ty\| - 1.$$

$$\overline{\delta}_X(t) = \inf_{x \in S_X} \sup_{\dim(X/Y)<\infty} \inf_{y \in S_Y} \|x + ty\| - 1.$$

More concretely. $\overline{\rho}_X(t)$ is the best (smallest) constant such that for all $x \in S_X$ and all $(x_\alpha)_\alpha$ weakly null net in B_X, $\limsup \|x + tx_\alpha\| \leq 1 + \overline{\rho}_X(t)$.

- The norm is *asymptotically uniformly smooth* (AUS) if $\lim_{t \to 0} \frac{\overline{\rho}_X(t)}{t} = 0$.
- It is *asymptotically uniformly convex* (AUC) if $\overline{\delta}_X(t) > 0$, for all $t > 0$.
- The norm of X is p-AUS if : $\exists C > 0 \ \ \forall t > 0 \ \ \overline{\rho}_X(t) \leq Ct^p$ $(1 < p < \infty)$.
- The norm is *asymptotically uniformly flat* (AUF) if there exists $t_0 > 0$ such that $\overline{\rho}_X(t) = 0$, for all $t \leq t_0$.
- The norm of X is q-AUC if : $\exists C > 0 \ \ \forall t \in (0, 1] \ \ \overline{\delta}_X(t) \geq Ct^q$.

Examples. let $(F_n)_n$ be a sequence of finite dimensional normed spaces. Then $(\sum_{n=1}^{\infty} F_n)_{\ell_p}$ is p-AUS and p-AUC, and $(\sum_{n=1}^{\infty} F_n)_{c_0}$ is AUF.

Duality between asymptotic smoothness and convexity

Let $(X, \| \ \|)$ be a Banach space and $t \in [0, \infty)$.

$$\overline{\delta}_X^*(t) = \inf_{x^* \in S_{X^*}} \sup_{Y \in \mathcal{E}} \inf_{y^* \in S_Y} \|x^* + ty^*\| - 1,$$

where $\mathcal{E}$ is the set of weak*-closed finite codimensional subspaces of X^*.

Concretely. $\overline{\delta}_X^*(t)$ is the best (largest) constant such that for all $x^* \in S_X$ and all $(x_\alpha^*)_\alpha$ weak* null net in S_{X^*}, $\liminf \|x^* + tx_\alpha^*\| \geq 1 + \overline{\delta}_X^*(t)$.

- The norm of X^* is *weak* asymptotically uniformly convex* (AUC*) if $\overline{\delta}_X^*(t) > 0$, for all $t > 0$.
- The norm of X^* is q-AUC* if : $\exists C > 0 \ \ \forall t \in (0, 1] \ \ \overline{\delta}_X(t) \geq Ct^q$.

Proposition

Let X be a Banach space, $p \in (1, \infty]$ and q be the conjugate of p. Then
(i) $\| \ \|_X$ is AUS iff $\| \ \|_{X^*}$ is AUC*.
(ii) $\| \ \|_X$ is p-AUS iff $\| \ \|_{X^*}$ is q-AUC*.

Two useful results.

Notation. For a Banach space X, we denote $X \in \langle P \rangle$ if X admits an equivalent norm with property P.

Johnson, Lindenstrauss, Preiss, Schechtman (2002)

Let $p \in (1, \infty)$ and X be a Banach space. Then TFAE
(i) X is separable reflexive and $X \in \langle p - AUS \rangle \cap \langle p - AUC \rangle$.
(ii) There exists a sequence of finite dimensional normed spaces $(F_n)_n$ such that X is isomorphic to a subspace of $(\sum_{n=1}^{\infty} F_n)_{\ell_p}$.

Godefroy, Kalton, L. (2000)

Let X be a Banach space. Then TFAE
(i) X is separable and $X \in \langle AUF \rangle$.
(ii) X is isomorphic to a subspace of c_0.

The Szlenk index.

Let X be a Banach space and K be a weak* compact subset of X^*. For each $\varepsilon > 0$, define

$$s_\varepsilon(K) = \{x^* \in K,\ \forall V\ w^* - \text{neighborhood of } x^*,\ \text{diam } (V \cap K) \geq \varepsilon\}.$$

Given an ordinal α, $s_\varepsilon^\alpha(K)$ is defined inductively by letting $s_\varepsilon^0(K) = s_\varepsilon(K)$, $s_\varepsilon^{\alpha+1}(K) = s_\varepsilon(s_\varepsilon^\alpha(K))$ and $s_\varepsilon^\alpha(K) = \cap_{\beta<\alpha} s_\varepsilon^\beta(K)$ if α is a limit ordinal. We then define $\mathrm{Sz}(X,\varepsilon)$ as the least ordinal α so that $s_\varepsilon^\alpha(B_{X^*}) = \emptyset$, if such ordinal exists, and $\mathrm{Sz}(X,\varepsilon) = \infty$ otherwise. The *Szlenk index of X* is defined as

$$\mathrm{Sz}(X) = \sup_{\varepsilon>0} \mathrm{Sz}(X,\varepsilon).$$

Proposition

Let X be a separable Banach space. Then TFAE

(i) X^* is separable.

(ii) $\mathrm{Sz}(X) < \omega_1$ (where ω_1 is the first uncountable ordinal).

(iii) $Id : (B_{X^*}, w^*) \to (B_{X^*}, \|\ \|)$ is of first Baire class.

$\mathrm{Sz}(X)$ can be seen as a measure of how close to be non separable X^* is.

Proposition

(1) Let X be an AUS Banach space. Then $\mathrm{Sz}(X) \leq \omega$ (where ω is the first uncountable ordinal). More precisely, there exists $C \geq 1$ (universal) so that $\mathrm{Sz}(X,\varepsilon) \leq C(\overline{\delta}_X^*(\varepsilon/C))^{-1}$.

(2) Let $p \in (1,\infty]$ and q be the conjugate of p. If X is p-AUS then, there exists $C \geq 1$ such that $\sum_{i=1}^n \varepsilon_i^q \leq C$, whenever $s_{\varepsilon_1} \dots s_{\varepsilon_n}(B_{X^*}) \neq \emptyset$. We say that X has a *q-summable Szlenk index*.

Proof.

Knaust, Odell, Schlumprecht (1999)

Let X be a separable Banach space such that $\mathrm{Sz}(X) \leq \omega$. Then there exists $p \in (1,\infty]$ such that $X \in \langle p - AUS \rangle$.

M. Raja (2010) : extension to the non separable setting.

Godefroy, Kalton, L. (2001) : Let X be a separable Banach space. If $\mathrm{Sz}(X,\varepsilon) \leq C\varepsilon^{-q}$, then $X \in \langle r - AUS \rangle$, for all $r \in (1,p)$.

Asymptotic two players games

Notation. W_X denotes the set of all weak open neighborhood of 0 in the Banach space X and $cof(X)$ the set of all its closed finite codimensional subspaces. Fix $1 < p \leq \infty$, $c \geq 1$ and $n \in \mathbb{N}$.

The $T(c,p)$ game. It is a game with infinitely many rounds.
Round 1 : P_I chooses $U_1 \in W_x$, P_{II} chooses $x_1 \in U_1 \cap B_X$.
Round 2 : P_I chooses $U_2 \in W_x$, P_{II} chooses $x_2 \in U_2 \cap B_X$ ... and so on...
P_I wins if : $\forall a \in c_{00}$ $\|\sum_{i=1}^{\infty} a_i x_i\| \leq c\|a\|_p$.

Definition

$t_p(X) = \inf\{c > 0,\ P_I \text{ has a winning strategy in } T(c,p)\}$
$X \in T_p$ if $t_p(X) < \infty$.

The $A(c,p,n)$ game. Same game with n rounds.
P_I wins if : $\forall a \in \ell_p^n$ $\|\sum_{i=1}^{n} a_i x_i\| \leq c\|a\|_p$.

Definition

$a_p(X) = \sup_{n\in\mathbb{N}} \inf\{c > 0,\ P_I \text{ has a winning strategy in } A(c,p,n)\}$
$X \in A_p$ if $a_p(X) < \infty$.

The $N(c, p, n)$ game. Same game with n rounds.
P_I wins if : $\| \sum_{i=1}^{n} x_i \| \leq c n^{1/p}$.

Definition

$n_p(X) = \sup_{n \in \mathbb{N}} \inf\{c > 0,\ P_I \text{ has a winning strategy in } N(c, p, n)\}$
$X \in N_p$ if $n_p(X) < \infty$.

Remark. These games are determined.

Weakly null trees

Notation. Let$D$ be a set, $n \in \mathbb{N}$.
$D^{\leq n} = \{\emptyset\} \cup \bigcup_{i=1}^{n} D^i,\ D^{<\omega} = \bigcup_{n=0}^{\infty} D^{\leq n},\ D^{\omega} = D^{\mathbb{N}}$.
For $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_m)$, $s \frown t = (s_1, \ldots, s_n, t_1, \ldots, t_m)$.
$|s|$ is the length of s and for $i \leq |s|$, $s_{|_i} = (s_1, \ldots, s_i)$. Finally $s \preceq t$ if t extends s.

Definition. Let X be a Banach space and D be a weak neighborhood basis of 0 in X. Then $(x_t)_{t \in D^{<\omega}}$ is a *weakly null tree* in X if for all $t \in D^{<\omega}$, $(x_{t \frown (U)})_{U \in D}$ is a weakly null net, where D is directed by reverse inclusion.
If X^* is separable, we can take $D = \mathbb{N}$.

Proposition

A Banach space X belongs to T_p if and only if there exists $c > 0$ such that for any weak neighborhood basis of 0 in X and any $(x_t)_{t \in D^{<\omega}}$ weakly null tree in B_X, there exists $\tau \in D^\omega$ such that

$$\forall a \in c_{00} \quad \Big\| \sum_{i=1}^{\infty} a_i x_{\tau|_i} \Big\| \leq c \|a\|_p.$$

There are similar statements for A_p and N_p.

Inclusions

Theorem (R. Causey)

Let $1 < p < \infty$. Then $T_p \subsetneq A_p \subsetneq N_p \subsetneq \bigcap_{1<r<p} T_r$.
For $p = \infty$. $T_\infty \subsetneq A_\infty = N_\infty \subsetneq \bigcap_{1<r<\infty} T_r$.

Fundamental example. The Tsirelson space T^* belongs to $A_\infty \setminus T_\infty$.
Note. The class $A_\infty = N_\infty$ is the class of asymptotic-c_0 spaces.

Characterizations of T_p, A_p

Causey (2018)

Let X be a Banach space, $p \in (1, \infty]$, and q be its conjugate. Then TFAE
(i) $X \in T_p$.
(ii) $X \in \langle p - AUS \rangle$.
(iii) X admits an equivalent norm whose dual norm is q-AUC*.

Causey, Fovelle, L. (2023)

Let X be a Banach space, $p \in (1, \infty]$, and q be its conjugate. Then TFAE
(i) $X \in A_p$.
(ii) There exists $M \geq 1$ and $c > 0$ such that for all $\theta \in (0, 1]$, there exists a norm $|\ |$ on X satisfying

$$M^{-1}|\ | \leq \|\ \|_X \leq M|\ | \text{ and } \forall t \geq \theta,\ \overline{\rho}_{|\ |}(t) \leq ct^p.$$

(iii) X has a q-summable Szlenk index.

<u>**Note.**</u> If $\mathrm{Sz}(X) \leq \omega$, then there exists $q \in [1, \infty)$ and $C > 0$ such that $\mathrm{Sz}(X, \varepsilon) \leq C\varepsilon^{-q}$. Thus, for all $r < p$ (conjugate of q), $X \in \langle r - AUS \rangle$.

CHAPTER III.

SZLENK INDEX AND INTERLACING GRAPHS

Back to Kalton's interlacing graphs and Property Q_p

Braga, Petitjean, Procházka, L. (2023)

Let $p \in (1, \infty]$ and assume that X is a Banach space with A_p. Then X^* has property $\mathcal{Q}_p$.

Corollary

If the family $([\mathbb{N}]^k, d_I)_k$ equi-Lipschitz embeds into X^*. Then $\mathrm{Sz}(X) \geq \omega^2$.

Comments : It is a new obstruction to coarse Lipschitz embeddings in some AUC Banach spaces (AUC* duals in fact) which is an alternative to the approximate midpoint principle. It can give more precise quantitative information, but only in the non reflexive setting.

• Duals of spaces in A_∞ provide new examples of spaces with $\mathcal{Q}_\infty$. There are non reflexive separable spaces in $A_\infty \setminus T_\infty$.

Problem 5

Does any dual of an asymptotic-c_0 space coarsely embed into a reflexive Banach space ?

Proof

Assume $f : ([\mathbb{N}]^k, d_I) \to X^*$ is 1-Lipschitz.
By passing to an infinite subset of $\mathbb{N}$, we may assume that there exists a weak*-null tree $(x^*(\overline{m}))_{\overline{m}\in[\mathbb{N}]^{\leq k}}$ in X^* such that

1. (i) For all $\overline{n} \in [\mathbb{N}]^k$, $f(\overline{n}) = \sum_{\overline{m}\preceq\overline{n}} x^*(\overline{m})$.
2. (ii) For all $\overline{m} \in [\mathbb{N}]^{\leq k} \setminus \{\emptyset\}$, $\|x^*_{\overline{m}}\| \leq \mathrm{Lip}(f) \leq 1$.

$f(n_1,\dots,n_k) \underset{n_k\to\infty}{\to} \partial f(n_1,\dots,n_{k-1}) \underset{n_{k-1}\to\infty}{\to} \partial^2 f(n_1,\dots,n_{k-2})$..

Then set $x^*(n_1,\dots,n_k) = f(n_1,\dots,n_k) - \partial f(n_1,\dots,n_{k-1}),\dots,$
$x^*(n_1) = \partial^{k-1} f(n_1) - \partial^k f$.

By passing to a further infinite subset of $\mathbb{N}$, we "may assume" that

- (a) $\forall i \leq k\ \exists \varepsilon_i \in [0,1]\ \forall \overline{m} \in [\mathbb{N}]^i\ \ \|x(\overline{m})\| \approx \varepsilon_i$ (Ramsey).
- (b) $\|x^*(2n_1,\dots,2n_i) - x^*(2n_1+1,\dots,2n_i+1)\| \gtrsim \varepsilon_i$.

Set now, for $\overline{n} = (n_1,\dots,n_i)$, with $i \leq k$:

$$y^*(\overline{n}) = x^*(2n_1,\dots,2n_i) - x^*(2n_1+1,\dots,2n_i+1)$$

$$\text{and}\quad z^*(\overline{n}) = \sum_{\overline{m}\preceq\overline{n}} y^*(\overline{m}).$$

Then $(z^*(\overline{n}))_{\overline{n}\in[\mathbb{N}]^{\leq k}}$ is a weak*-continuous tree ($y^*_\emptyset = z^*_\emptyset = 0$).

Most importantly $(z^*(\overline{n}))_{\overline{n}\in[\mathbb{N}]^{\leq k}} \subset B_{X^*}$. Indeed, for $\overline{n} \in [\mathbb{N}]^k$,

$$\|z^*(\overline{n})\| = \|f(2n_1, \dots, 2n_k) - f(2n_1+1, \dots, 2n_k+1)\| \leq \mathrm{Lip}\,(f) \leq 1.$$

The weak* null tree $(z^*(\overline{n}))_{\overline{n}\in[\mathbb{N}]^{\leq k}}$ testifies that $s_{\varepsilon_1} \dots s_{\varepsilon_k}(B_{X^*}) \neq \emptyset$.
But $X \in A_p$ and thus has a q-summable Szlenk index (q conjugate of p).
So there exists $C \geq 1$ (depending on X) such that $\sum_{i=1}^k \varepsilon_i^q \leq C$.
Applying Hölder, we get $\sum_{i=1}^k \varepsilon_i \leq C^{1/q}k^{1/p}$ and, for all $\overline{n}, \overline{m} \in [\mathbb{N}]^k$:

$$\|f(\overline{n}) - f(\overline{m})\| = \|\sum_{\emptyset \prec \overline{v} \preceq \overline{n}} x^*(\overline{v}) - \sum_{\emptyset \prec \overline{u} \preceq \overline{m}} x^*(\overline{u})\| \leq 2C^{1/q}k^{1/p}.$$

Application

Let $p \in (1,\infty)$. The James space $\mathcal{J}_p$ has $\mathcal{Q}_r$ if and only if $r \leq q$ (conjugate of p) and it has HC_s if and only if $s \leq p$.
In particular $\mathcal{J}_p$ does not coarse Lipschitz embed into $\mathcal{J}_{p'}$, for $p \neq p'$.

Comments. It is customary, in order to use AUC as an obstruction to coarse Lipschitz embeddings, to apply the approximate midpoints principle. Concentration properties say more : they give information on the so-called compression exponents.

Optimality : Lipschitz free spaces again !

BLPP (2023)

There exists a separable Banach space X such that $\mathrm{Sz}(X) = \omega^2$ and $M = ([\mathbb{N}]^{<\omega}, d_I)$ Lipschitz embeds into X^*.

Idea of proof. We will show that $\mathcal{F}(M) \simeq Y \subseteq X^*$, with $\mathrm{Sz}(X) = \omega^2$.

- Let (s_n) be the summing basis of c_0. Define $f(n_1, \dots, n_k) = \sum_{i=1}^k s_{n_i}$. Let $M_k = f([\mathbb{N}]^{\leq k})$ (which is 2-Lipschitz equivalent to $([\mathbb{N}]^{\leq k}, d_I)$). Finally, set P_k to be the weak*-closure of M_k in ℓ_∞.

- $P_k = \Big\{ \sum_{i=1}^j s_{n_i} + \ell \mathbb{1} \ :\ j, \ell \in \mathbb{N} \cup \{0\},\ j + \ell \leq k,\ n_1 < \ldots < n_j \in \mathbb{N} \Big\}$.

- P_k is bounded, countable, uniformly discrete and w*-compact. Then (Garcia-Lirola, Petitjean, Procházka, Rueda-Zoca - 2018), $\mathcal{F}(P_k)$ is isometric to X_k^*, where X_k is the space $Lip_0(P_k) \cap C_{w^*}(P_k)$ equipped with the Lipschitz norm.

- We appeal now to another paper of Kalton (2004) to say that $\mathcal{F}(M)$ linearly embeds into $(\sum_k \mathcal{F}(M_{2^k}))_{\ell_1}$, and therefore in $(\sum_k \mathcal{F}(P_{2^k}))_{\ell_1}$, which is the dual of $X = (\sum_k X_{2^k})_{c_0}$.

- Since P_k is bounded and uniformly discrete, X_k is isomorphic to a subspace of $(C_{w^*}(P_k), \| \|_\infty)$. The Cantor-Bendixon index of (P_k, w^*) is finite (exercise). Thus X_k is isomorphic to a subspace of c_0 (Bessaga-Pełczyński).
- We conclude that $\mathrm{Sz}(X_k) = \omega$ and therefore that $\mathrm{Sz}(X) \leq \omega^2$.

Embeddings with extra assumptions.

Kalton

If $([\mathbb{N}]^k, d_I)_k$ equi-coarsely embed into a separable Banach lattice X, then X contains a subspace isomorphic to c_0.

BLPP

If c_0 coarse Lipschitz embeds into a dual space X^* with coarse Lipschitz distortion strictly less than $\frac{3}{2}$, then X contains an isomorphic copy of ℓ_1.

BLPP

Neither c_0 nor L_1 can be coarsely embedded into a separable dual Banach space by a map that is weak-to-weak* sequentially continuous.

CHAPTER IV.

THE GORELIK PRINCIPLE

AND ASYMPTOTIC UNIFORM SMOOTHNESS

Gorelik (1994) ; Johnson, Lindenstrauss, Schechtman (1996)

Let $p \in (1, \infty)$. If a Banach space X is uniformly homeomorphic to ℓ_p, then it is linearly isomorphic to ℓ_p.

Proposition 1

Let X be a Banach space, $X_0 \in cof(X)$ and $0 < c < d$. Then there exists a compact subset A of dB_X such that for all $\Phi : A \to X$ continuous so that $\|\Phi(a) - a\| \leq c$ for all $a \in A$, we have : $\Phi(A) \cap X_0 \neq \emptyset$.

Proof. Let $Q : X \to X/X_0$ quotient map. There exists $L : X/X_0 \to X$ continuous st $QL = Id_{X/X_o}$ and $A = L(cB_{X/X_0}) \subset dB_X$ (Bartle-Grave). $dim(X/X_0) < \infty$, so A is compact and let $\Phi : A \to X$ as in the statement. For $y \in cB_{X/X_0}$, set $g(y) = y - (Q \circ \phi \circ L)(y)$. Then, for all $y \in cB_{X/X_0}$

$$\|g(y)\| = \|(Q \circ L)(y) - (Q \circ \phi \circ L)(y)\| \leq \|L(y) - (\phi \circ L)(y)\| \leq c.$$

So $g : cB_{X/X_0} \to cB_{X/X_0}$ being continuous, it follows from Brouwer's theorem, that there exists $y \in B_{X/X_0}$ such that $g(y) = y$, i.e. $\Phi(L(y)) \in X_0$. □

The Gorelik Principle

Let X, Y Banach spaces; $f : X \to Y$ homeomorphism such that $\mathrm{Lip}(f^{-1}) < M$. Then, for any $X_0 \in cof(X)$ and any $\lambda > 0$, there exists a compact subset K of Y so that

$$\lambda B_Y \subset K + f(2M\lambda B_{X_0}).$$

Proof. Let $c = \lambda \mathrm{Lip}(f^{-1}) < d = \lambda M$ and $A \subset dB_X$ be given by Prop. 1.
Fix $y \in \lambda B_Y$ and define $\Phi : A \to X$ by $\Phi(a) = f^{-1}(y + f(a))$.
Then Φ is continuous and $\|\Phi(a) - a\| \leq \lambda \mathrm{Lip}(f^{-1}) = c$, $a \in A$.
By Prop. 1, there exists $a \in A$ such that $f^{-1}(y + f(a)) \in X_0$.
But in fact, $f^{-1}(y + f(a)) \in 2\lambda M B_{X_0}$.
Therefore, $y \in -f(a) + f(2\lambda M B_{X_0})$. Set $K = -f(A)$, compact in Y, to conclude the proof. □

Comment. A weak* null sequence $(y_n^*)_n$ in Y^* is normed by $f(2MB_{X_0})$ and therefore by $f(x_n)$ with (x_n) weakly null in X (assuming X^*, Y^* separable)... almost as well as it is normed by a weakly null sequence in Y. This intuitively explains why we will deduce results on the preservation of AUC* (or equivalently AUS).

Definition.

Let X, Y be Banach spaces. A map $f : X \to Y$ is a coarse Lipschitz equivalence if f is coarse Lipschitz and there exist $g : Y \to X$ coarse Lipschitz and $C > 0$ such that

$$\forall x \in X \ \ \|g \circ f(x) - x\| \leq C \ \text{ and } \ \forall y \in X \ \ \|f \circ g(y) - y\| \leq C.$$

We denote $X \overset{CL}{\sim} Y$.

Note. A uniform homeomorphism is a coarse Lipschitz equivalence, but there are Banach spaces that are coarse Lipschitz equivalent but not uniformly homeomorphic (Kalton).

A variant of the Gorelik Principle

Let X, Y Banach spaces such that $X \overset{CL}{\sim} Y$. Then there exist $C > 0$, $M \geq 1$ and $\lambda_0 > 0$ such that for all $X_0 \in cof(X)$ and all $\lambda \geq \lambda_0$ there exists a compact subset K of Y such that

$$\lambda B_Y \subset K + CB_Y + f(2\lambda M B_{X_0}).$$

The main applications

Godefroy, Kalton, L. (2000-2001)

Let $p \in (1, \infty]$. Then the class T_p is stable under Lipschitz equivalences.

Corollary (GKL)

Let X be a Banach space. If $X \overset{L}{\sim} c_0$, then $X \simeq c_0$.

Proof. Let $X \overset{L}{\sim} c_0$. Then $X \in T_\infty$ and is separable. Thus X is isomorphic to a subspace of c_0. By a result of Heinrich and Mankiewicz it is also a $\mathcal{L}^\infty$ space. Finally, a $\mathcal{L}^\infty$ infinite dimensional subspace of c_0 is isomorphic to c_0 (Johnson-Zippin).

GKL

Let $p \in (1, \infty]$. Then $\bigcap_{1<r<p} T_r$ is stable under uniform homeomorphisms.

Kalton (2013)

Let $p \in (1, \infty)$. The class T_p is not stable under uniform homeomorphisms.

Causey, Fovelle, L. (2023)

Let $p \in (1, \infty]$. Then the classes A_p and N_p are stable under coarse Lipschitz equivalences.

Problem 6

Is T_∞ stable under coarse Lipschitz equivalences ?
Does $X \underset{CL}{\sim} c_0$ imply $X \simeq c_0$?

Proof of the stability of T_∞ under Lipschitz equivalences

Let X, Y Banach spaces and assume that $f : X \overset{L}{\sim} Y$ and $X \in T_\infty$. We may assume that X is separable and thus that $X \subseteq c_0$; that $\mathrm{Lip}(f) = 1$ and $\mathrm{Lip}(f^{-1}) < M$. We will build a an equivalent norm on Y whose dual norm is 1-AUC* by letting

$$|y^*| = \sup\Big\{\frac{\langle y^*, f(x) - f(x')\rangle}{\|x - x'\|},\ x \neq x' \in X\Big\}.$$

Clearly $|\ |$ is equivalent to $\|\ \|_{Y^*}$ and w*-l.s.c., thus the dual of a equivalent norm on Y.

• Let $y^* \in Y^*$, $(y_k^*)_k \subset Y^*$ such that $y_k^* \overset{w^*}{\to} 0$ and $\|y_k^*\| \geq t$. We want to prove that $\liminf |y^* + y_k^*| \geq |y^*| + ct$, for some $c > 0$.
Pick $x \neq x' \in X$ so that $\langle y^*, f(x) - f(x')\rangle \approx \|x - x'\|\,|y^*|$.
We may assume $x' = -x$ and $f(x') = -f(x)$. So $\langle y^*, f(x)\rangle \approx \|x\|\,|y^*|$.
Since $X \subseteq c_0$, there exists $X_0 \in cof(X)$ such that

$$\forall z \in \|x\| B_{X_0}, \quad \|x + z\| = \|x - z\| \approx \|x\| \tag{1}$$

- By the Gorelik Principle, there exists a compact $K \subset Y$ such that

$$\frac{\|x\|}{2M} B_Y \subset K + f(\|x\| B_{X_0}).$$

Since $y_k^* \to 0$ uniformly on K, we can find $(z_k)_k \subset \|x\| B_{X_0}$ so that

$$\liminf_k \langle y_k^*, -f(z_k) \rangle \geq \frac{\|x\| t}{2M} \tag{2}$$

- Note that it follows from (1) that

$$\langle y^*, f(z_k) + f(x) \rangle = \langle y^*, f(z_k) - f(-x) \rangle \leq \|z_k - x\| \, |y^*| \lesssim \|x\| \, |y^*|.$$

So, $\langle y^*, f(z_k) \rangle \lesssim 0$. Since $y_k^* \overset{w^*}{\to} 0$ and by (2) :

$$\liminf_k \langle y^* + y_k^*, f(z_k) + f(x) \rangle \gtrsim \|x\| \, |y^*| + \frac{\|x\| t}{2M}$$

- Finally use again that $\|x - z_k\| \approx \|x\|$ and the definition of $| \ |$ to get

$$\liminf_k |y^* + y_k^*| \gtrsim |y^*| + \frac{t}{2M}.$$

$\square$

General statements on the preservation of AUS under equivalences

Godefroy, Kalton, L. (2001)

Let X and Y be Banach spaces such that there exists a Lipschitz equivalence f from X to Y. Then there exist a a universal constant $K > 0$ and a constant $M > 0$ (depending on f) so that there exists a norm $|\ |$ on Y satisfying

$$\forall y \in Y,\ \|y\|_Y \le |y| \le M\|y\|_Y \ \text{ and } \ \forall t \in [0,1],\ \overline{\rho}_{|\ |}(K^{-1}M^{-2}t) \le \overline{\rho}_X(t).$$

Dalet, L. (2017)

Let X and Y be Banach spaces such that there exists a coarse Lipschitz equivalence f from X to Y. Then there exist a a universal constant $K > 0$ and a constant $M > 0$ (depending on f) so that for any $\varepsilon > 0$, there exists a norm $|\ |$ on Y satisfying

$$\forall y \in Y,\ \|y\|_Y \le |y| \le M\|y\|_Y \ \text{ and }$$

$$\forall t \in [0,1],\ \overline{\rho}_{|\ |}(K^{-1}M^{-2}t) \le \overline{\rho}_X(t) + \varepsilon.$$

Bates, Johnson, Lindenstrauss, Preiss, Schechtman (1999)

Being Asplund (and in particular having a separable dual) is stable under Lipschitz equivalences (even Lipschitz quotients).

Dutrieux (2001)

There exists $\psi : (0,\omega_1) \to (0,\omega_1)$ so that $\mathrm{Sz}(Y) \leq \psi(\mathrm{Sz}(X))$, whenever X^* separable and $X \overset{L}{\sim} Y$.

Problem 7

$\psi = Id$? Is the Szlenk index preserved by Lipschitz equivalences ?

We know $\psi(\omega) = \omega$. This would imply a positive answer to the following :

Problem 8

Let K, L be compact metric spaces.
Does $C(K) \overset{L}{\sim} C(L)$ imply $C(K) \simeq C(L)$?

Warning. It is not true for uniform homeomorphisms !

Ribe (1984)

Let $(p_n)_n \subset (1, \infty)$ be a decreasing sequence so that $\lim_n p_n = 1$ and let $X = (\sum_{n=1}^{\infty} \ell_{p_n})_{\ell_2}$. Then X is uniformly homeomorphic to $Y = X \oplus \ell_1$.

Remarks. X is reflexive, while Y is not Asplund.
Quantitatively : $\mathrm{Sz}(X) = \omega^2$, while $\mathrm{Sz}(Y) = \infty$.

THANK YOU