From property (β) of Rolewicz to metric geometry

In 1987, Stefan Rolewicz introduced a geometric property of the norm of a Banach space in relation to well-posedness of optimization problems, called the property (β).

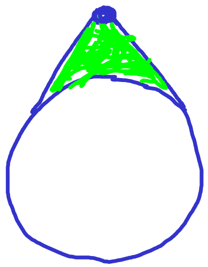
Kuratowski measure of noncompactness of a set A in a metric space

$$\alpha(A) = \inf \{ \varepsilon > 0 : \exists \text{ finite cover } A \subseteq \bigcup_{i=1}^{n} A_i, \ \operatorname{diam}(A_i) < \varepsilon \}$$

Let X be a Banach space with a closed unit ball B.

Let $x \in X$, $\|x\| > 1$.

drop $D(x,B) = \operatorname{conv}(x, B)$, $R(x,B) = D(x,B) \setminus B$



$\|\cdot\|$ has property (β) if $\forall\, \varepsilon > 0$
$\exists\, \delta > 0$ s.t. $1 < \|x\| < 1+\delta \Rightarrow \alpha(R(x,B)) < \varepsilon$

(i.e. $\alpha(R(x,B)) \to 0$ uniformly in x, $\|x\| \to 1$)

[R] $UC \Rightarrow (\beta) \Rightarrow NUC \Rightarrow$ reflexivity

(Huff)

$\not\Leftarrow$ $\not\Leftarrow$ isometrically

question about isomorphically

$\not\Leftarrow$ [Montesinos-Torregrosa] [K]

$\not\Leftarrow$ [K]

new isomorphic class between superreflexive and reflexive

Asymptotic moduli [Milman]

named by [Johnson - Lindenstrauss - Preiss - Schechtman]

Let X be a Banach space, $t > 0$.

AUC modulus $$\overline{\delta}_X(t) = \inf_{\|x\|=1} \sup_{\operatorname{codim}(Y)<\infty} \inf_{\substack{\|y\|=1 \\ y \in Y}} (\|x+ty\|-1)$$

$\|\cdot\|$ is AUC if $\forall\, t > 0$, $\overline{\delta}(t) > 0$.

AUS modulus $$\overline{\rho}_X(t) = \sup_{\|x\|=1} \inf_{\operatorname{codim}(Y)<\infty} \sup_{\substack{\|y\|=1 \\ y \in Y}} (\|x+ty\|-1)$$

NUC = AUC + reflexivity

NUS = AUS + reflexivity introduced by Prus

[P] characterized NUC and NUS isomorphically for spaces with bases in terms of (p,q)-estimates 1989

[K1] 1991 isometric characterization

$\|\cdot\|$ has property (β) if $\forall\ \varepsilon>0\ \exists\ \delta>0$ s.t. $\forall\ \|x\|<1$

and (x_n), $\|x_n\|\leq 1$ with $\mathrm{sep}((x_n)) = \inf\{\|x_n - x_m\| : n\neq m\} > \varepsilon$,

$\exists\ n_k$ s.t. $\|(x + x_{n_k})/2\| < 1-\delta$.

same time [K2] isometrically $(\beta) \not\Rightarrow$ NUS, but $\exists$ equiv. norm with (β)

$\Leftrightarrow$ $\exists$ eq. NUC + $\exists$ eq. NUS

(for spaces with a basis)

[Dilworth - K. - Lancien - Randrianarivony] 2017

true for all Banach spaces

uses [K2], [Knaust - Odell - Schlumprecht], Szlenk index

---

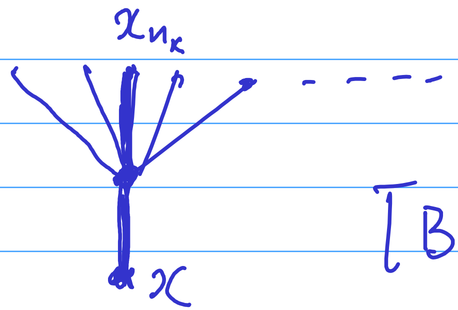
back to isometric characterization. Take $(-x_n)$

$\forall\ \varepsilon>0\ \exists\ \delta>0$ s.t. $\forall\ x\in B$, $\forall\ (x_n)\subseteq B$

with $\inf\{\mathrm{dist}(x_n, x_m) : n\neq m\} > \varepsilon$, $\exists\ n_k$ s.t. $\mathrm{dist}(x, x_{n_k}) < 2(1-\delta)$

metric

[Lima-Randrianarivony] 2012 used this definition to answer a 10-y. old question of [Bates-J-L-P-S] from metric geometry: a Banach space which is a uniform quotient of ℓ_p, $1<p<2$ is linearly isomorphic to a linear quotient of ℓ_p. For $p>2$ [JLPS] and [Mendel-Naor] (Markov convexity)



umbel

[Baudier - Gartland] umbel convexity

(X,d_X), (Y,d_Y) metric spaces. $f: X \to Y$ is a bi-Lip embedding if $\exists\ s>0$ (scaling) and $\exists\ D \geq 1$ (distortion factor)

$$s \cdot d_X(x,y) \leq d_Y(f(x),f(y)) \leq D \cdot s \cdot d_X(x,y)$$

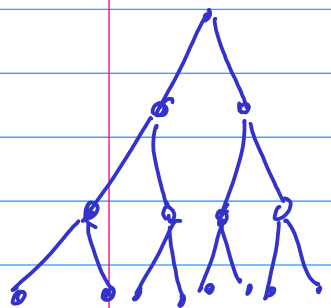
A natural way to understand the geometry of a metric space is to study in which metric spaces, in particular Banach spaces, it does or does not bi-Lipschizly embed. (Also the converse point of view).

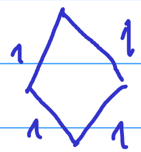
Ribe program

[Bourgain] 1986 B. space X is <u>not</u> superreflexive $\Longleftrightarrow$ binary trees of any height uniformly bi-Lip. embed in X

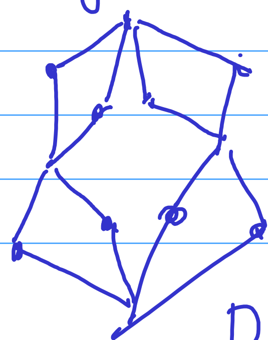
[Johnson - Schechtman] 2009

binary diamonds, Laakso graphs





D_1



D_2

[Ostrovskii - Randrianantoanina]

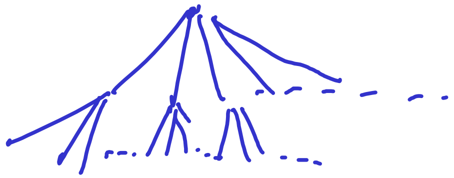
k - branching diamonds

[Swift] generated by bundle graph

Less clear asymptotic Ribe program.

[Baudier-Kalton-Lancien] 2010 proved in particular for a reflexive space X, one can bi-Lip. embed countably branching trees $\Longleftrightarrow$ X has no equiv. AUC norm or X has no eq. AUS norm.

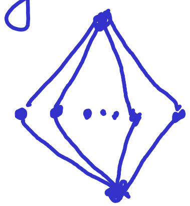
Thus, X has equiv. norm with property (β) $\Longleftrightarrow$ one can not bi-Lip. embed count. branching trees.



asymptotic superreflexive spaces

[BKL] extended by Yoël Perreau to quasi-reflexive spaces.

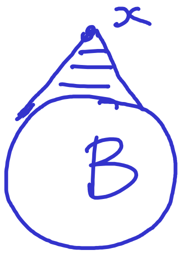
What about characterizing (β) by countably branching diamonds?



[Revalski - Zhivkov] 2012 compact UC (again, coming from optimizations)

= (isometrically) (β)

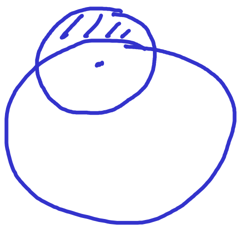
[DKRRZ1]



$\operatorname{diam}(R(x)) \to 0 \quad = \quad UC$

unif. in $\|x\| \to 1$

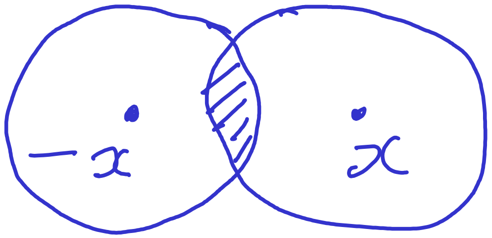
$\alpha(R(x)) \to 0 \quad = (\beta)$



$\operatorname{diam}() \to 0 \quad = UC$

$\alpha() \to 0 \quad = \beta$

what about this $\operatorname{lens}(x,\delta) = B[x, 1+\delta] \cap B[-x, 1+\delta]$



$\operatorname{diam}(\operatorname{lens}(x,\delta)) \to 0 \quad = UC$

uniformly in $\delta \to 0$

(*) $\alpha(\operatorname{lens}(x,\delta)) \to 0 \quad = (\beta)$?

answer: NO [DKRRZ2]

Definition. AMUC = (*)

asymptotic midpoint UC

AMUC modulus of $(X, \|\cdot\|)$. Let $t>0$

$$\tilde{\delta}_X(t) = \inf_{\|x\|=1} \sup_{\operatorname{codim}(Y)<\infty} \inf_{\|y\|=1} \left(\max\{\|x+ty\|, \|x-ty\|\} - 1 \right)$$

AMUC $\iff$ $\forall\, t>0$, $\tilde{\delta}_X(t) > 0$

Clearly, AUC $\Rightarrow$ AMUC

$\not\Leftarrow$ isometrically $\quad\Leftarrow$ isomorphically under some unconditionality type conditions

Question: Are they isomorphically equivalent?

[Baudier – Causey – Dilworth – K. – Randrianarivony – Schlumprecht – Zhang]

Theorem. If X has an equivalent AMUC norm, then countably branching diamonds (actually, more general graphs) do NOT uniformly bi-Lip embed in X.

Corollary. countably branching diamonds do not characterize asymptotic supperreflexive spaces (that is those with equivalent (β) norm)

Thank you!

Also, thanks to the organizers!
(in case I forgot to say that
in the beginning of my
talk)