# Convergence of remote projections onto convex sets

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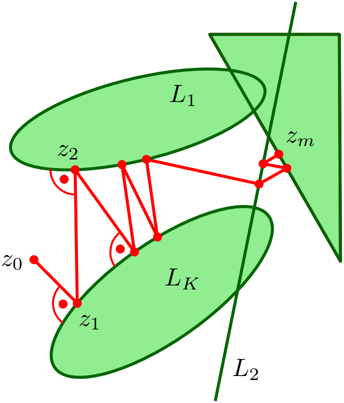
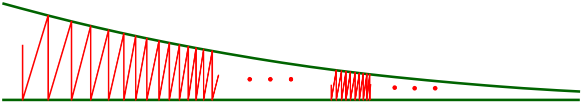
# DO THE ITERATES OF PROJECTIONS CONVERGE?



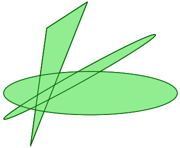

$K$ fixed, *e.g.* $K = 5$
$C_1, C_2, \dots, C_K \subset \mathbb{R}^d$ or $\ell_2$
closed convex sets

$k_1, k_2, \dots \in \{1, 2, \dots, K\}$ be arbitrary
$z_n = P_{k_n} z_{n-1}$ sequence of projections

Do the iterates converge to a point in the intersection $A = \bigcap C_k$, if it is non-empty?



# Convex sets



H Hilbert space
closed and convex $C_1, C_2, \ldots, C_K \subset H$
$\bigcap C_i \neq \emptyset$

$z_n = P_{k_n} z_{n-1}$ iterates of the nearest point projections of a point $z$

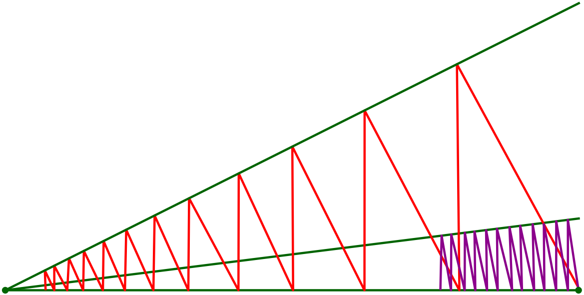
If $H = \mathbb{R}^d$ then $\{z_n\}$ converges.
[Dye, Kuczumow, Lin, Reich '96]

There exist 2 closed and convex sets $C, D \subset \ell_2$ with $0 \in C \cap D$, and a sequence $\{z_n\}$ of iterates of nearest point projections on these sets which does NOT converge in norm.
[Hundal '04]

There exist 3 closed subspaces $L_1, L_2, L_3 \subset \ell_2$ with the following property. For every $0 \neq z_0 \in H$ there is a sequence $k_1, k_2, \cdots \in \{1, 2, 3\}$ so that the sequence of iterates defined by $z_n = P_{k_n} z_{n-1}$ does not converge in norm.
[Eva Kopecká, Vladimír Müller, Adam Paszkiewicz, '14,'17]

# remotest projections onto symmetric convex sets converge

Let all $C_\alpha$, $\alpha \in \Omega$, be closed, convex, and symmetric subsets of a Hilbert space. (Hence $0 \in \bigcap C_\alpha$.) Then the sequence of remotest projections $z_n = P_{\alpha_n} z_{n-1}$ where
$\mathrm{dist}\,(z_n, C_{\alpha_n}) = \max_{\alpha\in\Omega} \mathrm{dist}\,(z_n, C_\alpha)$
converges in norm for any starting element $z_0 \in H$.
[Borodin, Kopecká '23]



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[Borodin, Kopecká '23]

relaxing symmetry by uniform almost symmetry:
$\forall \alpha : x \in C_\alpha \Rightarrow -\frac{1}{10} x \in C_\alpha$
Here we WLOG assume $0 \in C_\alpha \subset B(0,5)$ for all $\alpha$;
$\frac{1}{10}$ can be replaced by a $\delta \in (0,1]$.

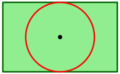
relaxing remoteness: choose $\alpha_n$ so that
$\mathrm{dist}\,(z_n, C_{\alpha_n}) \geq \frac{1}{2} \max_{\alpha\in\Omega} \mathrm{dist}\,(z_n, C_\alpha)$

More generally, take $t_n \in (0,1]$ instead of $\frac{1}{2}$
satisfying the condition
$\forall (a_i) \in \ell_2$ with $a_i \geq 0 :\ \liminf_{m\to\infty} \frac{a_m}{t_m} \sum_{i=1}^{m} a_i = 0$
invented by [Temlyakov '02] for all $C_\alpha$ hyperplanes; best possible.

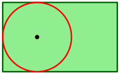
# degrees of symmetry

bounded, closed and convex $C, C_\alpha \subset H$ Hilbert space

$C$ is symmetric: $x \in C \Rightarrow -x \in C$

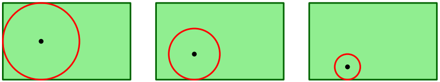


$C$ is somewhat symmetric: $\forall x \in C \, \exists \delta_x \in (0,1] : \; -\delta_x x \in C$



$C$ is almost symmetric: $\exists \delta \in (0,1] : \; x \in C \Rightarrow -\delta x \in C$

$C$ is somewhat symmetric $\Leftrightarrow$ $C$ is almost symmetric



The family $C_\alpha$, $\alpha \in \Omega$, is uniformly almost symmetric:

$\exists \delta \in (0,1] \, \forall \alpha : \; x \in C_\alpha \Rightarrow -\delta x \in C_\alpha$

# somewhat symmetric $\Leftrightarrow$ almost symmetric

bounded, closed and convex $C \subset H$ Hilbert space

$C$ is somewhat symmetric: $\forall x \in C \, \exists \delta_x \in (0,1] : \, -\delta_x x \in C$

$C$ is almost symmetric: $\exists \delta \in (0,1] : \, x \in C \Rightarrow -\delta x \in C$

$C$ is somewhat symmetric $\Leftrightarrow$ $C$ is almost symmetric

$\Leftarrow$: clear

Proof 1 $\Rightarrow$: Suppose $\inf_{x \in C} \delta_x = 0$.
Choose $x_n \in C$ having $\delta_{x_n} < 1/3^n$.
Then $x = \sum_{n=1}^{\infty} \frac{x_n}{2^n} \in C$, hence $\delta_x > 0$ implying after some computations $\delta_{x_k} > 1/3^k$ for large $k$'s - a contradiction.

Proof 2 $\Rightarrow$: (Baire category thm)
Both sets $C \cap (-C)$ and $\operatorname{conv}(C \cup (-C))$ generate norms on $\operatorname{span} C$ in which $\operatorname{span} C$ is a Banach space. These norms are equivalent in view of the open mapping theorem.
Hence $\inf_{x \in C} \delta_x > 0$.

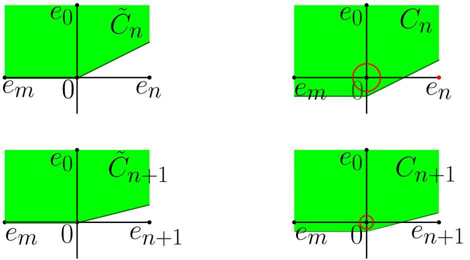
# UNIFORM almost symmetry needed

There are $C_n$, $n \in \mathbb{N}$, closed, convex, and (NOT uniformly!!) almost symmetric subsets of $\ell_2$ and a sequence $\{z_n\}$ of remotest projections onto these sets which does NOT converge in norm.

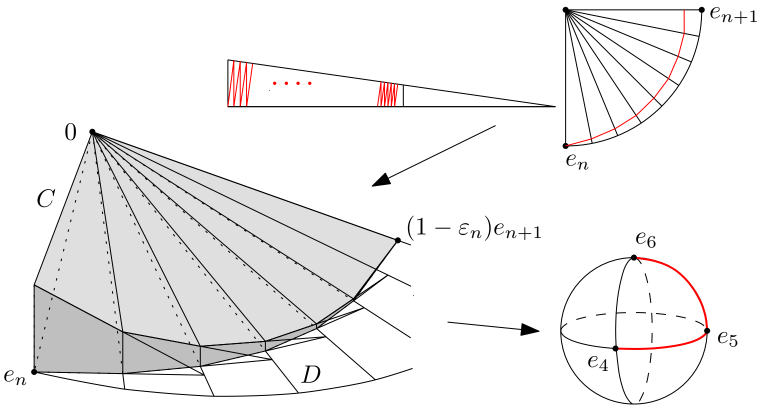
$e_0, e_1, e_2, \ldots$ ON basis of $\ell_2$, $D = e_0^{\perp}$, $a_n \searrow 0$, $\delta_n \downarrow 0$, $j_n \nearrow \infty$

$\phi_n : D \to [0, \infty)$ convex, continuous, positively homogeneous s.t.
$\phi_n(e_n) = a_n$, $\phi_n(-e_n) = \phi_n(\pm e_m) = 0$ if $m \neq n$

$\tilde{C}_n$ is the epigraph of $\phi_n$, $C = \bigcap \tilde{C}_n$, $C_n$ is the epigraph of $\phi_n - \delta_n$
$|(P_{\tilde{C}_n} P_D))^{j_n} e_n - e_{n+1}| < 5^{-n}$ and $|(P_{C_n} P_D)^{j_n} e_n - e_{n+1}| < 5^{-n}$



# No norm-convergence in $\ell_2$ already for 2 convex sets




In $\ell_2$ there exist a closed convex set $C$, a hyperplane D, with $0 \in C \cap D$, and a point $z$ so that the iterates $(P_C P_D)^n z$ do *not* converge in norm.
The iterates approximately contain an ON sequence $\{e_n\}$.

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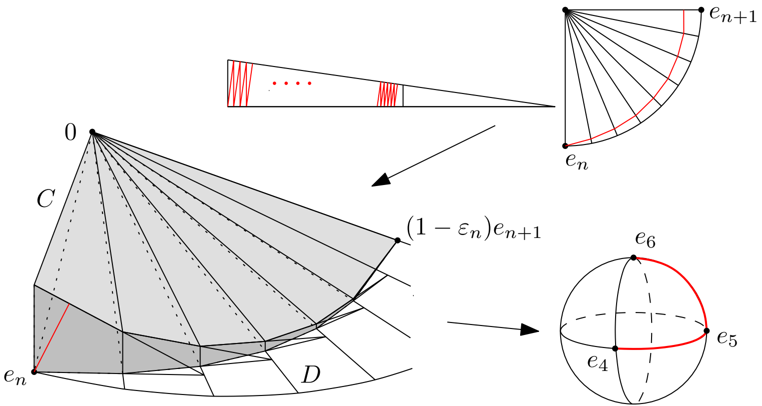


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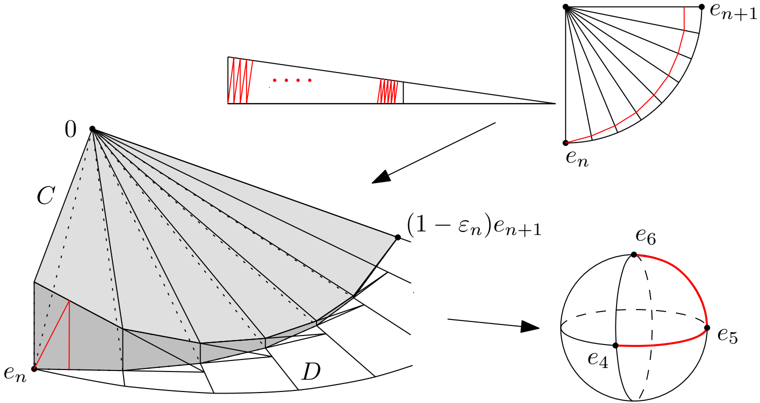


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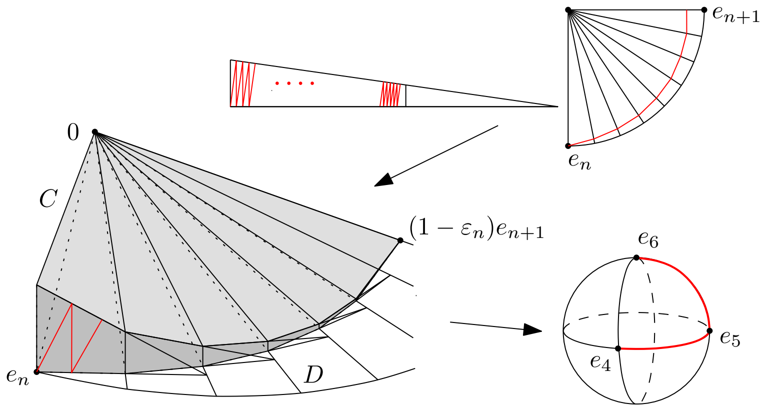


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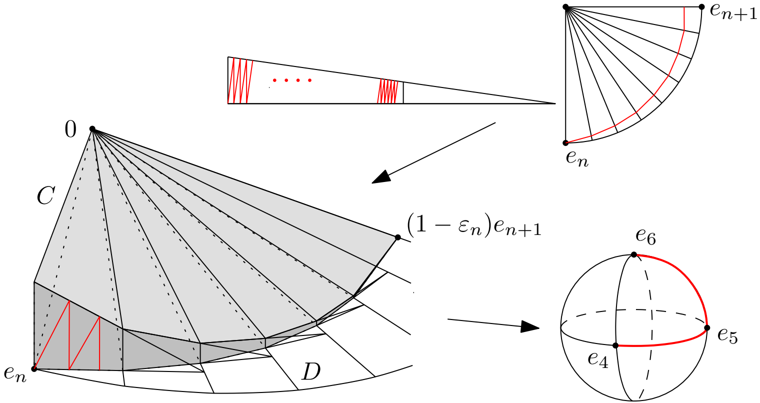


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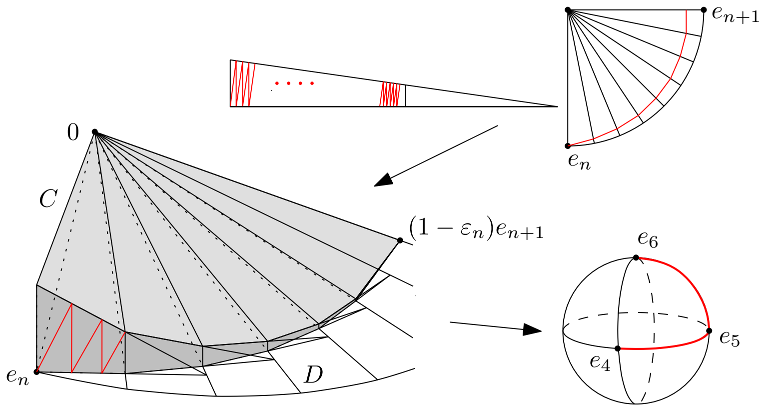


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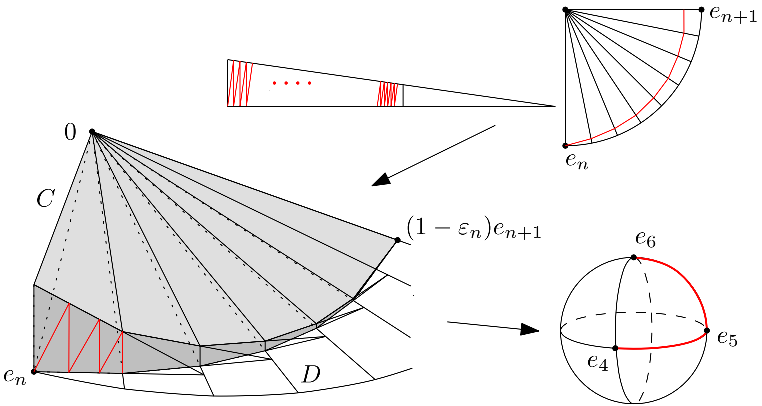


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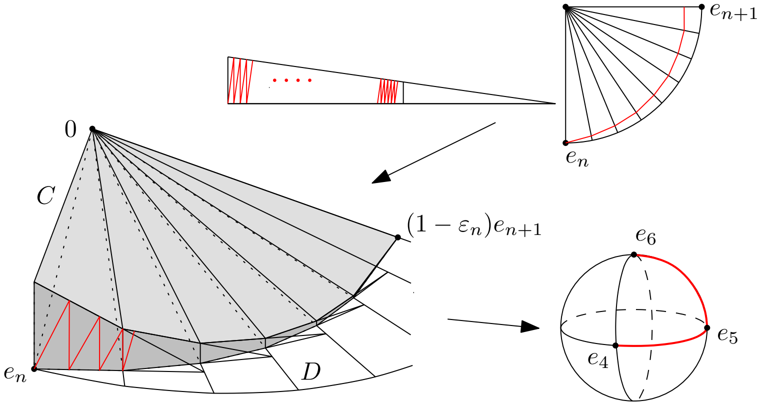


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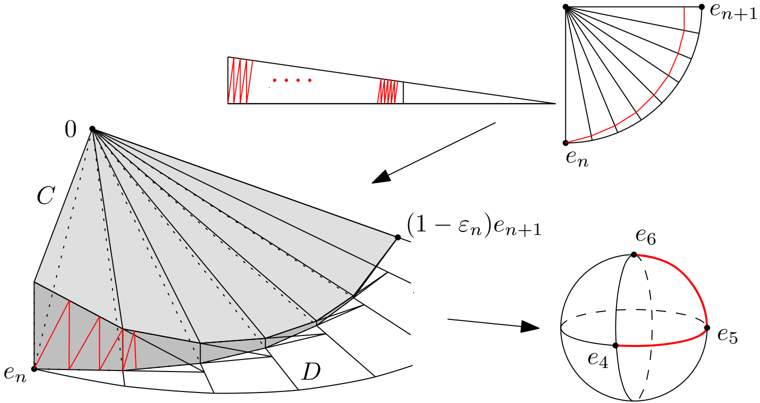


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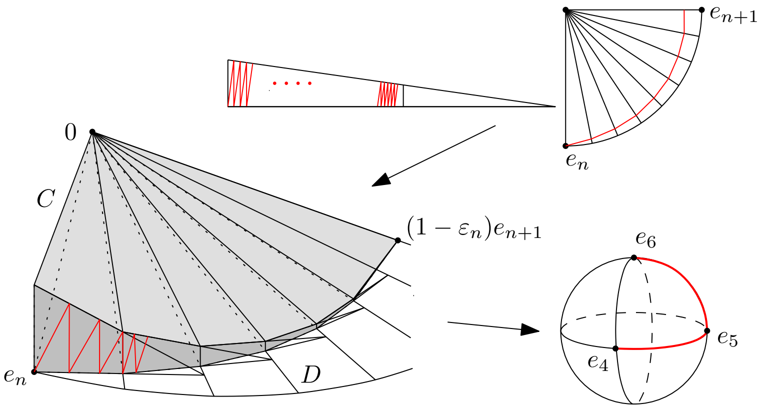


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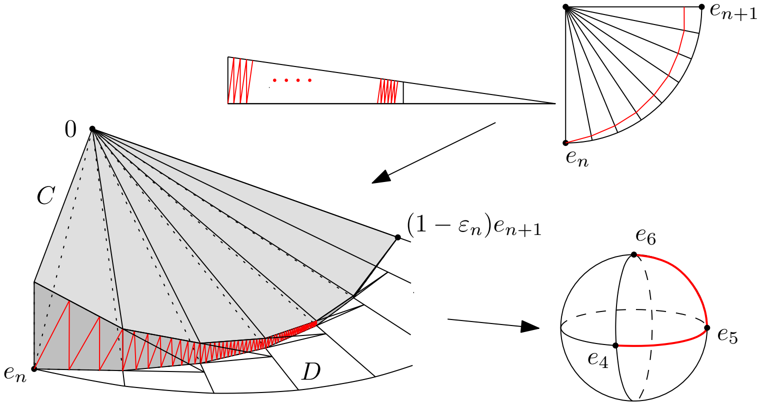


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