Riemann Integration and Asymptotic Structure of Banach Spaces

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Presentation outline

Riemann Integration

Let X be a Banach space. Riemann integrability of $f : [0,1] \to X$ is defined in exactly the same way as usual:

Definition

A function $f : [0,1] \to X$ is Riemann-integrable (RI) if $\exists\ x_f \in X$ such that $\forall\ \varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon) > 0$ such that

$$\left\| x_f - \sum \Delta_P(i) f(t_i) \right\| < \varepsilon$$

if P is a partition of $[0,1]$ such that $\pi(P) = \max \Delta_P(i) < \delta$.

Departure from familiar scalar situation if X is arbitrary Banach space: a RI f need not be Lebesgue almost everywhere (μ-a.e.) continuous (noted originally by L.M. Graves in 1927)!

Some Banach space theory definitions

A sequence (e_i) is said to be basic in X if, $\forall\ x \in \overline{\text{span}}\{e_i\} := [e_i]$, $\exists$ unique scalars (λ_i) such that $x = \sum_{i=1}^{\infty} \lambda_i e_i$. In particular, (e_i) is said to be a (Schauder) basis for X if $X = [e_i]$.

Now, X is said to be asymptotic-ℓ_p w.r.t. (e_i) if $\exists\ C > 0$ such that $\forall\ n \in \mathbb{N}$, $(x_i) \overset{C}{\sim} \ell_p^n$ if $x_1 < \ldots < x_n$ are normalized blocks of (e_i) with support at least n.

Also, a normalized basic sequence (e_i) generates a spreading model (v_i) of X if $\exists\ \delta_n \searrow 0$ so that $\left| \| \sum_{k=1}^{n} \lambda_k e_{i_k} \| - \| \sum_{i=1}^{n} \lambda_i v_i \| \right| < \delta_n$ $\forall\ |\lambda_i| \leq 1$ and $\forall\ n \leq i_1 < \ldots < i_n$. By Ramsey: every normalized basic sequence has a subsequence that generates a spreading model

An illustrative example

Fix $1 < p < \infty$, let (e_i) be the unit vector basis for ℓ_p, and let (r_i) be a listing of $\mathbb{Q} \cap [0,1]$. Then, the function $f : [0,1] \to \ell_p$ with $f(s) = 0$ for irrational s and with $f(r_i) = e_i$ satisfies:

$$\left\| \sum \Delta_P(i) f(t_i) \right\|_{\ell_p} = \left(\sum |\Delta_P(i) e_i^*[f(t_i)]|^p \right)^{\frac{1}{p}}$$

$$\leq \left(\sum \Delta_P(i)^{p-1} \Delta_P(i) \right)^{\frac{1}{p}} \leq \pi(P)^{\frac{p-1}{p}} < \varepsilon$$

for all fine enough partitions of $[0,1]$ so that f is RI (with integral zero). Note that:

- f is everywhere discontinuous on $[0,1]$
- Same construction works for c_0 but not for ℓ_1.

The Lebesgue Property

Consequently, there is the following definition...

Definition

A Banach space X has the Lebesgue Property (LP) if every RI function $f : [0, 1] \to X$ is μ-a.e. continuous.

Some remarks:

- LP is hereditary and isomorphic invariant
- Easy to show that $f : [0, 1] \to X$ is Darboux-integrable (DI) iff f is bounded and μ-a.e. continuous. Then, X has LP iff the collections of RI and DI functions $f : [0, 1] \to X$ coincide.

A bit of history ≈1980s

As noted, problem of determining which Banach spaces have the LP originated with Graves in 1927. Then, in the 1980s:

- da Rocha Filho & Pelczynski: Every spreading model of X is equivalent to ℓ_1 if X has the LP
  - Converse is true if $X \subset L_1[0,1]$ (original proof of this was lost)
- da Rocha Filho: ℓ_1 and Tsirelson space T have the LP
- Pizzotti: many results (including characterizing the LP) in her thesis - in Portuguese

A bit of history ≥ 1990

The results of da Rocha Filho and Pizzotti seem to have been published only locally in Brazil and gone unnoticed in general.

- Gordon: survey paper where he re-proves that ℓ_1 and T have LP. He credits da Rocha Filho but was apparently never made aware of Pizzotti's thesis (not cited)
- Rodriguez: lists characterization of LP as open problem in survey paper of open problems in Banach space theory
- Naralenkov/Gaebler: X has LP if it is (coordinate-free) asymptotic-ℓ_1

A helpful lemma

After some cursory speculation about the converse result, the more natural question that Bunyamin and I asked is:

- Does X contain an asymptotic-ℓ_1 subspace if it has LP?

It turns out that the answer is no (as we will see), but pursuing this question led us to a lemma of Pizzotti that we found in the thesis of G. Martinez-Cervantes.

Let D be the set of dyadic rational numbers in $(0, 1)$ with the natural enumeration (d_j). Now:

Theorem

Suppose that X does not have the LP. Then, $\exists$ a normalized basic sequence (x_j) such that the function $f : [0, 1] \to X$ that is defined by $f(d_j) = x_j$ and by $f(s) = 0$ if $s \notin D$ is RI.

The LP is a sequential condition

Not difficult to show that $f : [0,1] \to X$ is RI iff $\exists\, x_f \in X$ such that

$$\forall\, \varepsilon > 0,\ \exists\, n = n(\varepsilon) \in \mathbb{N} \text{ so that } \left\| x_f - \frac{1}{2^m} \sum_{i=1}^{2^m} f(t_i) \right\| < \varepsilon$$

for every $m \geq n$ and for every choice $t_i \in \left(\frac{i-1}{2^m}, \frac{i}{2^m}\right)$ of interior tags. Then, in view of the Pizzotti lemma, X has the LP if, for every normalized basic sequence (x_j) and function $f : [0,1] \to X$ defined by $f(d_j) = x_j$ and $f(s) = 0$ if $s \notin D$, we have that:

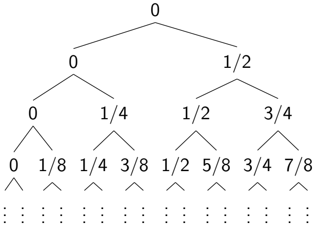
$$\exists\, c_f > 0 \text{ so that } \forall\, n \in \mathbb{N},\ \left\| \frac{1}{2^m} \sum_{i=1}^{2^m} f(t_i) \right\| \geq c_f$$

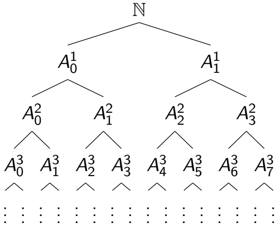
for some $m \geq n$ and interior tags $t_i \in \left(\frac{i-1}{2^n}, \frac{i}{2^n}\right)$.

- Conversely, what condition is implied for normalized basic sequences if X has the LP?

Haar systems and dyadic trees

Every dyadic subinterval is associated (by a subsequence (x_{j_k})) to infinite subset of $\mathbb{N}$ and we should find "bad tags" in these subsets with arbitrarily large support. We therefore define the dyadic trees





where $(A_j^n)_{j=0,n\in\mathbb{N}}^{2^n-1}$ is a fixed collection of infinite subsets of $\mathbb{N}$ (a "Haar system") such that $\mathbb{N} = \bigcup_{j=0}^{2^n-1} A_j^n$ and $A_j^n = A_{2j}^{n+1} \cup A_{2j+1}^{n+1}$. Then, we map $\frac{j}{2^n}$ from the left tree to the $(2^n)^{\text{th}}$ member of A_j^n from the right tree.

The function $g(d_j) = e_{\sigma(d_j)}$

Note that this map, σ, takes $\frac{l}{2^m} \in \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right)$ to the $(2^m)^{\text{th}}$ member of $A_l^m \subset A_{j-1}^n$. Then, if X has the LP and (e_i) is a normalized basic sequence, the function $g(d_j) = e_{\sigma(d_j)}$ with $g(s) = 0$ if $s \notin D$ is not RI, and it follows that

$$\exists\ c_g > 0 \text{ s.t. } \forall\ n \in \mathbb{N},\ \left\| \frac{1}{2^m} \sum_{k=0}^{2^m-1} g(t_k) \right\| = \left\| \frac{1}{2^m} \sum_{k=0}^{2^m-1} e_{\sigma(d_{j_k})} \right\| \geq c_g$$

for some $m \geq n$ and (assumed for now to be) dyadic interior tags $t_k \in \left(\frac{k-1}{2^m}, \frac{k}{2^m}\right)$. Note that $\sigma(d_{j_k})$ is, by construction, at least the $(2^m)^{\text{th}}$ member of A_k^m, and (e_i) therefore satisfies the condition:

$$c_g \leq \lim_{n\to\infty} \sup \left\{ \frac{1}{2^m} \left\| \sum_{k=0}^{2^m-1} e_{i_k} \right\| \;\middle|\; m \geq n \text{ and } 2^m \leq i_k \in A_k^m \right\}$$

Characterization of the LP

This observation motivates:

Definition

A normalized basic sequence (e_i) in X is said to be Haar-ℓ_1^+ if, for every Haar system of partitions $(A_j^n)_{j=0,n\in\mathbb{N}}^{2^n-1}$ of $\mathbb{N}$, there exists a constant $C > 0$ such that

$$C \leq \lim_{n\to\infty} \sup \left\{ \frac{1}{2^m} \left\| \sum_{j=0}^{2^m-1} e_{i_j} \right\| \;\middle|\; m \geq n \text{ and } 2^m \leq i_j \in A_j^m \right\}$$

Note the order of quantifiers: a fixed Haar system determines g as above which, in turn, determines the constant $C > 0$. Then,

Theorem

X has the LP iff every normalized basic sequence in X is Haar-ℓ_1^+.

Relation between Haar-ℓ_1^+ and asymptotic structures

Recall that every spreading model of X is equivalent to ℓ_1 if X has the LP. What about other local asymptotic structures?

Asymptotic models are array generalizations of spreading models and it is not difficult to show that the LP falls between the notions of ℓ_1 spreading and asymptotic models in the following sense:

$$\ell_1 \text{ asymptotic models} \implies \text{Haar-}\ell_1^+ \text{ (LP)} \implies \ell_1 \text{ spreading models}$$

However, neither converse implication holds.

- There exist Schur spaces that fail LP
- X_{iw} due to Argyros and Motakis has LP, but every infinite-dimensional subspace has a c_0 asymptotic model

Final thoughts

The X_{iw} example shows that a Banach space with the LP need not even contain an asymptotic-ℓ_1 subspace.

- Not surprising because the condition of being asymptotic-ℓ_1 is even stronger than having ℓ_1 asymptotic models

But there is an obvious question that we did not answer yet:

- If every spreading model of X is equivalent to ℓ_1, then does X contain an infinite-dimensional subspace that has the LP?

After some discussion with Motakis, we believe that the answer is no by a non-trivial modification of X_{iw}.

Thank you!