On isometry groups on the space of converging sequences

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Group actions

Throughout the talk, G is a fixed group.

A linear action $\lambda : G \curvearrowright Z$ of G on a Banach space Z, i.e. a group homomorphism

$$\lambda : G \to GL(Z) \subseteq B(Z)$$

will always be assumed bounded, i.e. $\sup_{g \in G} \|\lambda(g)\| < +\infty$.

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Given λ as above, a subspace $X \subseteq Z$ is G-invariant if $\lambda(g)(X) = X, \forall g \in G$.

Definition

Fix $\lambda : G \curvearrowright Z$ as above, and a G-invariant subspace $X \subseteq Z$.
Then G acts on X, by $u(g)x := \lambda(g)_{|X}x$
and on $Y := Z/X$, by $v(g)(z + X) := \lambda(g)z + X$.

Objective

Study relations between u, v, λ. In particular, given u, v, what are the possible λ's (we call them compatible with u, v)?

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- Kalton "centralizers" ($\geq$ 1979): $G =$ *units* in a Banach lattice
- Pytlic-Szwarc, Ozawa, Pisier... ($\geq$ 1986): $X, Y = \ell_2, Z = \ell_2 \oplus \ell_2$, $u = v$ a unitary action.
- F. - Rosendal (2017): $Z = X \oplus Y$
- UEx + USP ($\geq$ 2017): complex structures $G = \{1, i, -1, -i\}$
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Let us concentrate on the case $Z = X \oplus Y$.

Examples of compatible λ's on $X \oplus Y$

On the blackboard:

- diagonal actions,
- conjugate actions,
- c_0 and c.

Derivations

Observation

Given X, Y and u, v as above, the following are equivalent

1. λ_d is a compatible bounded action on $X \oplus Y$, where

$$\lambda_d(g) := \begin{pmatrix} u(g) & d(g) \\ 0 & v(g) \end{pmatrix},$$

2. $d : G \to B(Y, X)$ is a bounded map such that $\forall g, h \in G$, $d(gh) = u(g)d(h) + d(g)v(h)$.

Definition

A map d is a derivation (for u, v on X, Y) if it satisfies (1)(2) above.

Inner derivations

The simplest example of derivation

Given any $A \in B(Y, X)$, the formula

$$d_A(g) = [u(g), A, v(g)] =: u(g)A - Av(g)$$

defines a derivation, called inner.

Immediate

A derivation d is inner if and only if λ_d is conjugate to the diagonal representation.

Classical example: unitarizable actions

Assume G is countable, $X = Y = \ell_2(G)$, (with basis $(e_g)_{g \in G}$) and $u = v$ the regular unitary action on $\ell_2(G)$ defined by

$$u(g)(e_h) = e_{gh}.$$

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Theorems

- Day (50) and Dixmier (50): G is amenable $\Rightarrow$ all derivations for u, u are inner ($\Leftrightarrow$ all actions λ on $\ell_2(G) \oplus \ell_2(G)$ compatible with u, u are conjugate to the diagonal (unitary) action)

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Comments (linear unbounded maps, bounded non-linear maps)

A theorem

Let $0 \to X \to Z \to Y \to 0$, and actions u, v of a group G on X, Y respectively.

Theorem (Castillo, F. 23)

Assume G is amenable and X is a "G-ultrassummand" (for example X reflexive). Then:

- all actions of G on Z compatible with u and v (attention: there could exist no such actions) are mutually conjugate.

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- In particular, when $Z = X \oplus Y$, all such actions are conjugate to the diagonal action.

Pytlic-Szwarc's is a counterexample without amenability, let us see a counterexample without reflexivity: the already mentioned $G = 2^{<\omega}$ (amenable) acting on $c = c_0 \oplus \mathbb{R}$.

The counterexample of the group $2^{<\omega}$ acting on $c = c_0 \oplus \mathbb{R}$ (inspired by Antunes-F.-Grivaux-Rosendal (19))

We have $g = (\varepsilon_n)_n \in 2^{<\omega} = \{-1, 1\}^{<\omega}$, $X = c_0$, $Y = \mathbb{R}$, $Z = c$

$$u(g)(x) = (\varepsilon_n x_n)_n, \qquad v(g)(y) = y,$$

and

$$\lambda(g) = \begin{pmatrix} u(g) & d(g) \\ 0 & \mathrm{Id}_{\mathbb{R}} \end{pmatrix}.$$

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What is $d(g)$? Well, since $\lambda(g)(\underline{1}) = (\varepsilon_n)_n = \underline{1} - 2\sum_{\varepsilon_n=-1} e_n$,

$$d(g)y = -2y \sum_{\varepsilon=-1} e_n = -2p_{[e_n, \varepsilon_n=-1]}(y\underline{1})$$

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Is it inner? if $d(g) = [u(g), L, v(g)]$ holds for some L, then

$$d(g)y = u(g)Ly - Ly = (u(g) - Id_{c_0})Ly = -2p_{[e_n, \varepsilon_n=-1]}(Ly).$$

This holds $\forall g$ iff $Ly = y\underline{1}$, but this L does not belong to $B(\mathbb{R}, c_0)$.

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Why is it equivalent? If P is a G-equivariant projection from $Z = X \oplus Y$ onto X, then let $A : Y \to X$ be defined by $A(y) = P(0, y)$. Then

$$u(g)Ay = u(g)P(0, y) = P(\lambda(g)(0, y)) = P(d(g)y, v(g)y)$$

$$= P((d(g)y, 0)) + P(0, v(g)y) = d(g) + Av(g)y,$$

and therefore $d = d_A$ is inner. Conversely, given $d = d_A$, then $P(x, y) = x + Ay$ is G-equivariant.

THANK YOU - GRACIAS!

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