Generic Banach spaces

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based on joint work in progress with Marek Cúth and Noé de Rancourt

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Motivation

Let us motivate the topic of the talk by the following result.

Consider an infinite game of two players

I	E_1		E_3		$\dots$	
II		E_2		E_4		$\dots$

where players I and II alternate in playing finite-dimensional Banach spaces such that $E_i \subseteq E_{i+1}$, for every $i \in \mathbb{N}$.

After infinitely many steps they will have produced an increasing sequence $E_1 \subseteq E_2 \dots$ so that the completion

$$X := \overline{\bigcup_{i=1}^{\infty} E_i}$$

is a separable Banach space.

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Theorem [Kubiś (2018)]

There exists a unique, up to linear isometry, separable Banach space X (the Gurarij space for those who know what it is) such that Player II has a strategy whose following leads to X.

Motivation

We would like to introduce and describe Banach spaces admitting similar description as the Gurarij space.

Remark

In general, we must restrict the class of finite-dimensional Banach spaces the players are playing with. Suppose e.g. that Player I and II are allowed to play just with finite-dimensional Hilbert spaces. Then it is clear that Player II has a (very simple) strategy that leads to the infinite-dimensional separable Hilbert space as the result of the game.
Thus we already have two examples of separable Banach spaces from the class that we are interested in - the Gurarij space and the infinite-dimensional Hilbert space!

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2. **Topological characterization.** There is a Polish space of separable Banach space where Banach spaces X we are interested in satisfy:
   - The set of Banach spaces linearly isometric to X is G_δ.
   - In the (closed) set of Banach spaces finitely-representable in X, those that are linearly isometric to X form a comeager set (i.e. they are *generic*).

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2. **Topological characterization.** There is a Polish space of separable Banach space where Banach spaces X we are interested in satisfy:
   - The set of Banach spaces linearly isometric to X is G_δ.
   - In the (closed) set of Banach spaces finitely-representable in X, those that are linearly isometric to X form a comeager set (i.e. they are *generic*).
3. **Linear isometry group characterization.** Banach spaces X whose linear isometries are abundant in the following sense: For every finite-dimensional $E \subseteq X$ and every $\varepsilon > 0$ there are finite-dimensional $F \subseteq X$, $(1+\varepsilon)$-isometric embedding $\phi : E \to F$, and $\delta > 0$ such that for every $(1+\delta)$-isometric embedding $\psi : F \to X$ there is a linear isometry $T : X \to X$ such that $\|T \restriction E - \psi \circ \phi\| < \varepsilon$ (Diagram on blackboard).

Definition

The Gurarij space $\mathbb{G}$ is the unique separable Banach space that isometrically contains all finite-dimensional spaces and that satisfies that for every finite-dimensional $E \subseteq \mathbb{G}$, for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $(1+\delta)$-isometric embeddings $\phi, \psi : E \to \mathbb{G}$ there is a linear isometry $T : \mathbb{G} \to \mathbb{G}$ such that

$$\|\psi - T \circ \phi\| < 1 + \varepsilon.$$

Examples

Examples of 'generic' Banach spaces

1. the Gurarij space - follows from the game-characterization or from the definition and the third characterization.
2. $L_p([0,1])$, for $p \in [1,\infty)$ - this follows either from the work of Ferenczi, López-Abad, Mbombo, Todorčević on so-called Fraïssé Banach spaces, which satisfy the third description, or from our work Cúth-Doležal-D-Kurka, where it was computed that these spaces have G_δ isometry classes.
3. $L_p([0,1], L_q)$, for some p and q.

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There is also an abstract result implying the existence of certain generic Banach spaces which we, however, cannot identify with anything concrete.

Game characterization

For a Banach space X, denote by $\mathrm{Age}(X)$ the set (the isometry classes) of all finite-dimensional subspaces of X. By $\mathrm{Age}_{cl}(X)$ we denote the closure of $\mathrm{Age}(X)$ with respect to the Banach-Mazur distance.

Definition

Players I and II play an infinite game

I	E_1, ε_1		E_3, f_2, ε_3		$\ldots$	
II		E_2, f_1, ε_2		E_4, f_3, ε_4		$\ldots$

where we have $E_n \in \mathrm{Age}_{cl}(X)$, $\varepsilon_{n+1} < \varepsilon_n$ and $f_n : E_n \to E_{n+1}$ is a $(1 + \varepsilon_n)$-isometric embedding, for every $n \in \mathbb{N}$. Player II wins if $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and $\varinjlim(E_n, f_n)$ is isometric to the Banach space X, Player I wins otherwise.
(Note that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ implies $\prod \|f_n\| \leq \prod(1 + \varepsilon_n) < \infty$, so $\varinjlim(E_n, f_n)$ is defined.)

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X is generic if and only if Player II has a strategy in the game that leads to X.

The Polish space of separable Banach spaces

Recall that *Polish space* is a completely metrizable separable topological space (i.e. every separable Banach space is a Polish space). Every G_δ subset, in particular every closed subset, of a Polish space is Polish space itself. Also, a countable direct product of Polish spaces is a Polish space

The Polish space of separable Banach spaces

Denote by V the unique countable infinite-dimensional vector space over $\mathbb{Q}$. By a pseudonorm (or seminorm) we mean a non-negative valued function on a vector space that satisfies all the axioms of a norm except that it may vanish on non-zero elements. Denote by $\mathcal{P}$ the set of all pseudonorms on V. We can identify $\mathcal{P}$ with a closed subset of $\mathbb{R}^V$, thus by the remark above, it is a Polish space.

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Then we also define

- the space $\mathcal{P}_\infty \subseteq \mathcal{P}$ of pseudonorms λ such that the completion $\overline{(V,\lambda)}$ is infinite-dimensional;

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Then we also define

- the space $\mathcal{P}_\infty \subseteq \mathcal{P}$ of pseudonorms λ such that the completion $\overline{(V, \lambda)}$ is infinite-dimensional;
- the space $\mathcal{B} \subseteq \mathcal{P}_\infty$ of norms $\lambda \in \mathcal{P}_\infty$ such that the extension of λ to the completion $\overline{(V, \lambda)}$ remains a norm on the completion.

Both $\mathcal{P}_\infty$ and $\mathcal{B}$ are G_δ subspaces of $\mathcal{P}$, therefore they are Polish spaces.

Theorem (Cúth, Doležal, D., Kurka (2023))

- $L_2([0,1])$ is characterized as
  - the unique separable Banach space (up to isometry) whose isometry class is closed in $\mathcal{P}_\infty$ and $\mathcal{B}$;
  - the unique separable Banach space (up to isomorphism) whose isomorphism class is F_σ in $\mathcal{P}_\infty$ and $\mathcal{B}$.

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  - the unique separable Banach space (up to isomorphism) whose isomorphism class is F_σ in $\mathcal{P}_\infty$ and $\mathcal{B}$.
- The isometry classes of the Gurarij space and of $L_p([0,1])$, for $p \in [1,\infty) \setminus \{2\}$, are G_δ in $\mathcal{P}_\infty$ and $\mathcal{B}$ (and not F_σ).
- The isometry classes of ℓ_p, for $p \in [1,\infty) \setminus \{2\}$, are $F_{\sigma\delta}$ in $\mathcal{P}_\infty$ and $\mathcal{B}$ (and not 'simpler').

The Polish space of separable Banach spaces

Proposition (CDDK)

Let X be a separable infinite-dimensional Banach space and let $[X]$ be its isometry class in $\mathcal{B}$ (i.e. the set of norms defining isometrically X). Then the closure $\overline{[X]} \subseteq \mathcal{B}$ is the set of all separable infinite-dimensional Banach spaces (or rather norms defining them) that are finitely-representable in X.

In particular, if X is a separable infinite-dimensional Banach space with a G_δ isometry class, then in $\overline{[X]}$, the space of Banach spaces finitely-representable in X, X is dense G_δ - that is, generic.

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Since the space with a closed isometry class is unique and a G_δ isometry class is the second best thing (cf. the proposition above), we look for more examples and want to understand this class more.

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Remark

Spaces that have G_δ isometry class in $\mathcal{B}$ also have it in $\mathcal{P}$. It turns out that every finite-dimensional Banach space has a G_δ isometry class in $\mathcal{P}$, so finite-dimensional Banach spaces are generic!

Weak ultrahomogeneity

Definition

A (separable) Banach space X is *weakly ultrahomogeneous* if for every finite-dimensional $E \subseteq X$ and every $\varepsilon > 0$ there are finite-dimensional $F \subseteq X$, $(1+\varepsilon)$-isometric embedding $\phi : E \to F$, and $\delta > 0$ such that for every $(1+\delta)$-isometric embedding $\psi : F \to X$ there is a linear isometry $T : X \to X$ such that $\|T \upharpoonright E - \psi \circ \phi\| < \varepsilon$

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What is the advantage of this technical condition?

Remark

Notice that for separable Banach spaces X and Y that satisfy one of the three characterizations and such that $\mathrm{Age}_{cl}(X) = \mathrm{Age}_{cl}(Y)$ we can prove that $X \equiv Y$.
The most difficult verification is with the last - weak ultrahomogeneity - characterization.

Weak ultrahomogeneity

From the space to its Age

Let X be a separable infinite-dimensional weakly ultrahomogeneous Banach space. Then $\mathrm{Age}_{cl}(X)$ has the following properties:

1. $\mathrm{Age}_{cl}(X)$ contains spaces of arbitrarily large finite dimension;
2. if $F \in \mathrm{Age}_{cl}(X)$ and $E \subseteq F$ is a subspace, then $E \in \mathrm{Age}_{cl}(X)$;
3. if $E, F \in \mathrm{Age}_{cl}(X)$, then there exists $G \in \mathrm{Age}_{cl}(X)$ containing isometrically both E and F;

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3. if $E, F \in \mathrm{Age}_{cl}(X)$, then there exists $G \in \mathrm{Age}_{cl}(X)$ containing isometrically both E and F;
4. for every $E \in \mathrm{Age}_{cl}(X)$ and $\varepsilon > 0$ there are $F \in \mathrm{Age}_{cl}(X)$, $(1+\varepsilon)$-isometric embedding $\phi : E \to F$, and $\delta > 0$ such that for all $G, H \in \mathrm{Age}_{cl}(X)$ and $(1+\delta)$-isometric embeddings $\iota_G : F \to G$ and $\iota_H : F \to H$ there are $W \in \mathrm{Age}_{cl}(X)$ and isometric embeddings $\psi_G : G \to W$, $\psi_H : H \to W$ such that

$$\|\psi_H \circ \iota_H \circ \phi - \psi_G \circ \iota_G \circ \phi\| < \varepsilon.$$

(show a diagram on blackboard)

Weak ultrahomogeneity

From Age to a space

Let $\mathcal{K}$ be a class of finite-dimensional Banach spaces, closed with respect to the Banach-Mazur distance, satisfying the 4 properties from the previous slide.
Then there exists a unique separable infinite-dimensional weakly ultrahomogeneous Banach space X such that $\mathrm{Age}_{cl}(X) = \mathcal{K}$.

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Remark

For a separable Banach space X, sometimes it is difficult to verify that X is weakly ultrahomogeneous, while it is easier to check that $\mathrm{Age}_{cl}(X)$ satisfies the conditions 1-4: this is the case e.g. with $L_p([0,1], L_q)$.

Remark

There has been recently a (formally) strictly stronger notion of a (weak, cofinal) Fraïssé Banach space studied in

- Ferenczi, V., Lopez-Abad, J., Mbombo, B., and Todorcevic, S. *Amalgamation and Ramsey properties of L_p spaces*, Adv. Math. 369 (2020), 107190, 76.

We probably do not have an example of a generic Banach space that is not weakly cofinally Fraïssé in their sense.

Weak ultrahomogeneity

Theorem

Let X be a (infinite-dimensional) separable Banach space. TFAE:

1. X admits game-theoretic characterization from this talk.
2. The isometry class of X is comeager in the subspace of $\mathcal{B}$ consisting of spaces finitely representable in X.
3. The isometry class of X is G_δ.
4. X is weakly ultrahomogeneous.

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Remarks to the proof.

(1) $\Leftrightarrow$ (2): it follows from the Banach-Mazur game played in $\mathcal{B}$ and its re-interpretation.
(3) $\Rightarrow$ (2): This is clear.
(1) $\Rightarrow$ (4): We can prove the contraposition. If X is not weakly ultrahomogeneous, then it is possible to devise a strategy for Player I that prevents Player II from winning.
(4) $\Rightarrow$ (3): This is a tedious computation!

ω-categorical Banach spaces

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Definition/Theorem

Let X be a separable Banach space ad $\mathrm{LIso}(X)$ its linear isometry group. $\mathrm{LIso}(X)$ diagonally acts on B_X^n, for every $n \in \mathbb{N}$, and let us denote by $B_X^n /\!\!/ \mathrm{LIso}(X)$ the space of closures of orbits with the quotient topology. It is a complete metric space when equipped with the Hausdorff metric.

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X is *ω-categorical* if for each $n \in \mathbb{N}$, $B_X^n /\!\!/ \mathrm{LIso}(X)$ is compact.

ω-categorical Banach spaces

For every finite-dimensional, resp. separable, Banach space E, resp. X, and for every $C \geq 1$, denote by $\mathrm{Emb}_C(E, X)$ the space of all C-isometric embeddings of E into X with the operator-norm topology.

Fact

Let X be a separable Banach space. Then X is ω-categorical if and only if for every finite-dimensional E and $C \geq 1$, $\mathrm{Emb}_C(E, X) \mathbin{/\!\!/} \mathrm{LIso}(X)$ is compact.

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Let X be a separable Banach space. Then X is ω-categorical if and only if for every finite-dimensional E and $C \geq 1$, $\mathrm{Emb}_C(E, X) /\!\!/ \mathrm{LIso}(X)$ is compact.

This can be used to prove the very useful:

Proposition

Let X be a separable ω-categorical Banach space. Then every separable Banach space Y that is finitely representable in X isometrically embeds into X.

Call such a property *relative universality*.

Theorem

Let X be a separable infinite-dimensional ω-categorical Banach space. Then $\mathrm{Age}_{cl}(X) = \mathrm{Age}(X)$ satisfies the condition 1-4 above. In particular, there exists a separable infinite-dimensional weakly ultrahomogeneous Banach space Y such that $\mathrm{Age}(X) = \mathrm{Age}_{cl}(Y)$.

Theorem

Let X be a separable infinite-dimensional ω-categorical Banach space. Then $\mathrm{Age}_{cl}(X) = \mathrm{Age}(X)$ satisfies the condition 1-4 above. In particular, there exists a separable infinite-dimensional weakly ultrahomogeneous Banach space Y such that $\mathrm{Age}(X) = \mathrm{Age}_{cl}(Y)$.

In fact, we show that for every relatively universal separable infinite-dimensional Banach space X we have $\mathrm{Age}_{cl}(X) = \mathrm{Age}(X)$ satisfies the conditions 1-4.

Examples of ω-categorical Banach spaces

1. the infinite-dimensional separable Hilbert space (folklore and easy);
2. $L_p([0,1])$ for $p \in [1,\infty)$ (Ben Yaacov, Berenstein, Henson, Usvyatsov);
3. the Gurarij space (Ben Yaacov, Henson);
4. $C(2^{\mathbb{N}})$ (Henson);
5. $L_p([0,1], L_q)$ for $1 \leq p, q < \infty$ with $p \neq q$ (Henson, Raynaud).

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Remark

We do not know if every generic (weakly ultrahomogeneous) Banach space is ω-categorical. However, if X is ω-categorical, then the unique generic Banach space Y such that $\mathrm{Age}_{cl}(X) = \mathrm{Age}_{cl}(Y)$ is ω-categorical (so also $\mathrm{Age}_{cl}(Y) = \mathrm{Age}(Y)$).

$L_p(L_q)$ spaces

Proposition

Pick $p, q \in [1, \infty)$. Then the following conditions are equivalent

1. $L_p(L_q)$ is isometric to L_p.
2. $\mathrm{Age}_{cl}(L_p(L_q)) = \mathrm{Age}_{cl}(L_p)$.
3. $q = 2$, $p = q$ or $1 \leq p < q < 2$.

Idea

Clearly (1) imples (2).
We have

$$\mathrm{Age}_{cl}\big(L_p\big(L_q([0,1])\big)\big) = \overline{\bigcup_{n_1,\ldots,n_k \in \mathbb{N}, k \in \mathbb{N}} \mathrm{Age}(\ell_q^{n_1} \oplus_p \ldots \oplus_p \ell_q^{n_k})}.$$

This implies that (2) is equivalent with the fact that ℓ_q is finitely representable in L_p which is in turn equivalent to ℓ_q isometrically embeddable into L_p. This is known to hold if and only if (3) holds.

$L_p(L_q)$ spaces

Theorem

Let $p, q \in [1, \infty)$ such that $p \neq q$, $q \neq 2$, and we do not have $1 \leq p < q < 2$. Then $L_p(L_q)$ is (a new example of) a generic (weakly ultrahomogeneous) separable Banach space.

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Theorem

Let $p, q \in [1, \infty)$ such that $p \neq q$, $q \neq 2$, and we do not have $1 \leq p < q < 2$. Then $L_p(L_q)$ is (a new example of) a generic (weakly ultrahomogeneous) separable Banach space.

The proof relies on techniques of Mary-Angelica Tursi who proves that $L_p(L_q)$ is - in some sense - ultrahomogeneous as a Banach lattice (under the additional assumption that $p/q \notin \mathbb{N}$).

$L_p(L_q)$ spaces

Question

Let X be a separable generic Banach space and $p \in [1, \infty)$. Is $L_p(X)$ generic?

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Theorem

Let X be a separable Banach lattice such that $L_p(X)$ is ω-categorical. Let us denote by $\mathcal{F}$ the family of all the subspaces $\mathrm{span}\{\chi_{A_1}(t)x_1, \ldots, \chi_{A_l}(t)x_l\}$, where $A_1, \ldots, A_l$ are pairwise disjoint sets in $[0, 1]$ of positive measure and $x_1, \ldots, x_l$ points in X. Assume that the following two conditions hold

1. Given $F \in \mathcal{F}$, any isometric embedding $\psi : F \to X$ is disjoint preserving.
2. For any finite-dimensional $G \subset L_p(X)$ which is a sublattice, $\eta > 0$, and a lattice isometric embedding $\psi' : G \to X$ there is $T \in \mathrm{LIso}(X)$ satisfying $\| T \upharpoonright G - \psi' \| < \eta$.

Then $L_p(X)$ is generic.