Linearization of non-linear functions III

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Linearization of bounded holomorphic mappings

Let X and Y be complex Banach spaces. If $U \subset X$ is an open set, a mapping $f : U \to Y$ is said to be holomorphic if for every $x_0 \in U$ there exists a sequence $(P_k f(x_0))$, with each $P_k f(x_0)$ a continuous k-homogeneous polynomial, such that the series

$$f(x) = \sum_{k=0}^{\infty} P_k f(x_0)(x - x_0)$$

converges uniformly in some neighborhood of x_0 contained in U.

Equivalently, for every $x_0 \in U$, the function f is Fréchet differentiable at x_0; that is, there exists a differential of f at x_0, $df(x_0) \in \mathscr{L}(X, Y)$, such that

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - df(x_0)(h)}{\|h\|} = 0$$

Linearization of bounded holomorphic mappings

Some facts about holomorphic functions that we use in the sequel:

A function $f : U \to Y$ is said to be weakly holomorphic if $y^* \circ f$ is holomorphic, for all $y^* \in Y^*$.

Theorem

A function $f : U \to Y$ is holomorphic if and only if it is weakly holomorphic.

Let $f : B_X \to Y$ holomorphic and bounded: $\sup\{\|f(x)\| : x \in B_X\} < \infty$.

The differential of f at 0 is $df(0)(x) = \lim_{t\to 0} \frac{f(tx)-f(0)}{t}$.

As a consequence of Cauchy inequalities

$$\|df(0)\| \leq \sup_{x\in B_X} \|f(x)\|.$$

Linearization of bounded holomorphic mappings

Our next goal is the space

$$\mathscr{H}^\infty(B_X, Y) = \{f : B_X \to Y :\ f \text{ is holomorphic and bounded}\}$$

which is a Banach space with the norm $\|f\| = \sup_{x\in B_X} \|f(x)\|$.

A *linearization procedure* for this space was developed by Jorge Mujica in his article *Linearization of bounded holomorphic mappings on Banach spaces*, Trans. Amer. Math. Soc. (1991).

Linearization of bounded holomorphic mappings

The construction of the predual here is an abstract procedure that has been used in various situations. It is based on the following result:

MATH. SCAND. 29 (1971), 279–280

ON A THEOREM OF DIXMIER

KUNG-FU NG

A well-known theorem of Alaoglu (cf. [3, p. 84]) tells us that the closed unit ball in the Banach dual space of a normed space is compact with respect to the w^*-topology. In [1], Dixmier showed that this property is characteristic for Banach dual spaces. In this note, we shall give a short proof of a variant of Dixmier's theorem. This variant appears to be more convenient for applications [2]. Our argument is inspired by Edwards' paper [2] and is strictly elementary (in particular, we do not use the Krein–Smulian theorem).

THEOREM 1. *Let* $(X, \|\cdot\|)$ *be a normed space with closed unit ball* Σ. *Suppose there exists a (Hausdorff) locally convex topology* τ *for* X *such that* Σ *is* τ*-compact. Then* X *itself is a Banach dual space, that is, there exists a Banach space* V *such that* X *is isometrically isomorphic to the dual space* V' *of* V *(in particular,* X *is complete).*

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We consider τ_0 the compact-open topology on $\mathcal{H}^\infty(B_X)$. An application of Ascoli's theorem allows us to prove that $\overline{B}_{\mathcal{H}^\infty(B_X)}$ is τ_0-compact. In fact, on this ball, convergence in the topology τ_0 coincides with pointwise convergence.

By Dixmier-Ng theorem, $\mathcal{H}^\infty(B_X)$ is a dual space with predual given by

$$\mathcal{G}^\infty(B_X) = \{\varphi \in \mathcal{H}^\infty(B_X)^* : \varphi|_{\overline{B}_{\mathcal{H}^\infty(B_X)}} \text{ is } \tau_0 - \text{continuous}\}.$$

For $x \in B_X$ and $f \in \mathcal{H}^\infty(B_X)$ denote $\delta(x)(f) = f(x)$. Clearly $\delta(x) \colon \mathcal{H}^\infty(B_X) \to \mathbb{C}$ is linear and continuous meaning that $\delta(x) \in \mathcal{H}^\infty(B_X)^*$.

Also, $\delta(x)|_{\overline{B}_{\mathcal{H}^\infty(B)}}$ is τ_0-continuous so $\delta(x) \in \mathcal{G}^\infty(B_X)$.

We thus have $(\mathcal{G}^\infty(B_X))^* \cong \mathcal{H}^\infty(B_X)$.

Linearization of bounded holomorphic mappings

For $\mathcal{H}^\infty(B_X, Y) = \{f : X \to Y \text{ bounded holomorphic mappings}\}$ we have the Banach space $\mathcal{G}^\infty(B_X)$ and the mapping $\delta : X \to \mathcal{G}^\infty(B_X)$ given by $\delta(x)(f) = f(x)$ satisfying

- $\delta \in \mathcal{H}^\infty(B_X, \mathcal{G}^\infty(B_X))$ and $\|\delta\| = 1$. Since $f \circ \delta = f$ is holomorphic for each $f \in (\mathcal{G}^\infty(B_X))^* \cong \mathcal{H}^\infty(B_X)$ we have that δ is weakly holomorphic and hence holomorphic. Also,

$$\|\delta\| = \sup_{x \in B_X} \|\delta(x)\| = \sup_{x \in B_X,\, f \in B_{\mathcal{H}^\infty(B_X)}} |f(x)| = 1. \checkmark$$

- $\operatorname{span} \delta(B_X)$ is dense in $\mathcal{G}^\infty(B_X)$. It is evident that $f \in \mathcal{G}^\infty(B_X)^* \cong \mathcal{H}^\infty(B_X)$ satisfying $f|_{\delta(B_X)} \equiv 0$ should fulfill $f \equiv 0$. $\checkmark$
- For each $f \in \mathcal{H}^\infty(B_X, Y)$ there is a linear mapping $T_f \in \mathcal{L}(\mathcal{G}^\infty(B_X), Y)$ such that $f = T_f \circ \delta$. For $y^* \in Y^*$ we know that $y^* \circ f \in \mathcal{H}^\infty(B_X)$ with $\|y^* \circ f\| \leq \|y^*\| \|f\|$.

Linearization of bounded holomorphic mappings

Let us define

$$T_f : \mathscr{G}^\infty(B_X) \to Y^{**}$$
$$u \mapsto [y^* \mapsto \langle y^* \circ f, u\rangle].$$

It is clear that T_f is linear and $\|T_f\| \leq \|f\|$. Also, since $T_f(\delta(x)) = f(x) \in Y$ and $\operatorname{span} \delta(B_X)$ is dense in $\mathscr{G}^\infty(B_X)$ we obtain that $T_f(\mathscr{G}^\infty(B_X)) \subset Y$. ✓

- The mapping $\mathscr{H}^\infty(B_X, Y) \to \mathscr{L}(\mathscr{G}^\infty(B_X), Y)$ given by $f \mapsto T_f$ is a linear surjective isometry. It is easily seen that $f \mapsto T_f$ is linear. Also, for each $T \in \mathscr{L}(\mathscr{G}^\infty(B_X), Y)$ we obtain that $T \circ \delta \in \mathscr{H}^\infty(B_X, Y)$ with $\|T \circ \delta\| \leq \|T\|$. Thus, appealing to the previous bullet, the result holds. ✓

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Therefore, we have the commutative diagram and properties:

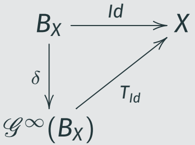
$$\begin{array}{ccc} B_X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle\delta}\big\downarrow & \nearrow_{T_f} & \\ \mathscr{G}^\infty(B_X) & & \end{array}$$

1. $\mathscr{G}^\infty(B_X)$ is unique (unless isometric isomorphism).
2. If $(f_i)_i$ is a bounded net in $\mathscr{H}^\infty(B_X)$ and $f \in \mathscr{H}^\infty(B_X)$ then $T_{f_i} \xrightarrow{w^*} T_f$ if and only if $f_i(x) \to f(x)$ for every $x \in X$.

Proposition

X is isometric to a 1-complemented subspace of $\mathscr{G}^\infty(B_X)$.

Proof. We *linearize* the mapping $Id \in \mathscr{H}^\infty(B_X, X)$:



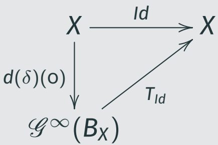
Now, we differentiate at 0 the identity $Id = T_{Id} \circ \delta$ obtaining

$$Id = d(Id)(0) = d(T_{Id})(\delta(0)) \circ d(\delta)(0) = T_{Id} \circ d(\delta)(0).$$

And the previous equality of linear mappings holds for every $x \in X$: $Id(x) = T_{Id}(d(\delta)(0)(x))$.

Linearization of bounded holomorphic mappings

This provides us with a new diagram



Let us see that $d(\delta)(0) : X \to \mathscr{G}^\infty(B_X)$ is an isometry.

On the one hand,

$$\|x\| = \|T_{Id}(d(\delta)(0)(x))\| \leq \|T_{Id}\| \|d(\delta)(0)(x)\| = \|d(\delta)(0)(x)\|.$$

On the other hand,

$$\begin{aligned} \|d(\delta)(0)(x)\| &= \sup_{f \in B_{\mathscr{H}^\infty(B_X)}} |\langle f, d(\delta)(0)(x) \rangle| = \sup_{f \in B_{\mathscr{H}^\infty(B_X)}} \|df(0)(x)\| \\ &\leq \sup_{f \in B_{\mathscr{H}^\infty(B_X)}} \|df(0)\| \|x\| \leq \|x\|. \end{aligned}$$

To see that $d(\delta)(0)(X)$ is 1-complemented in $\mathscr{G}^\infty(B_X)$ let us consider the mapping $Q = d(\delta)(0) \circ T_{Id} : \mathscr{G}^\infty(B_X) \to \mathscr{G}^\infty(B_X)$.

It is clear that $\|Q\| \leq 1$ and that this is a projection onto $d(\delta)(0)(X)$:

$$Q(d(\delta)(0)(x)) = d(\delta)(0) \circ T_{Id} \circ d(\delta)(0)(x) = d(\delta)(0) \circ Id(x) = d(\delta)(0)(x).$$

As in the linearization of polynomials we have:

Proposition

$\overline{B}_{\mathscr{G}^\infty(B_X)} = \overline{\Gamma}(\delta(B_X))$.

Proof. $\delta(B_X) \subset \overline{B}_{\mathscr{G}^\infty(B_X)}$ is a norming set for $\mathscr{H}^\infty(B_X)$.

Linearization of bounded holomorphic mappings

The bounded approximation property with constant 1 is called metric approximation property (MAP): X has the MAP if there exists a net of finite rank operators (T_α) such that $\|T_\alpha\| \leq 1$ and $T_\alpha(x) \to x$ for all $x \in X$.

Our goal is to prove: X has the MAP $\Leftrightarrow \mathscr{G}^\infty(B_X)$ has the MAP.

Each $f \in \mathscr{H}^\infty(B_X)$ is written as $f(x) = \sum_{k=0}^{\infty} P_k f(0)(x)$ for all $x \in B_X$. If we denote

$$S_m f(x) = \sum_{k=0}^{m} P_k f(0)(x) \quad \text{and} \quad \sigma_m f(x) = \frac{1}{m+1} \sum_{k=0}^{m} S_k f(0)(x)$$

we know that $S_m f(x) \to f(x)$ and $\sigma_m f(x) \to f(x)$.

With a clever argument Mujica proved that $\|\sigma_m f\| \leq \|f\|$.

Denoting by $\mathscr{P}(X, Y)$ the vector space of polynomials, it results:

Theorem

For each $f \in \overline{B}_{\mathscr{H}^\infty(B_X, Y)}$ we have a sequence $(\sigma_m f) \subset \overline{B}_{\mathscr{P}(X,Y)}$ such that $\sigma_m f(x) \to f(x)$ for every $x \in B_X$.

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$\mathscr{P}_f(X,Y)$: polynomials from X to Y whose images are contained in finite dimensional subspaces of Y.

Proposition

If X has the MAP, for each $f \in \overline{B}_{\mathscr{H}^\infty(B_X,Y)}$ there is a net $(P_\alpha) \subset \overline{B}_{\mathscr{P}_f(X,Y)}$ such that $P_\alpha(x) \to f(x)$ for every $x \in B_X$.

Proof. By the previous theorem it is enough to see that for $P \in \overline{B}_{\mathscr{P}(X,Y)}$ there is a net $(P_\alpha) \subset \overline{B}_{\mathscr{P}_f(X,Y)}$ such that $P_\alpha(x) \to P(x)$.

Since X has the MAP there is a net $(T_\alpha) \subset \overline{B}_{\mathscr{L}(X,X)}$ of finite rank operators converging point-wise to the identity. Taking $P_\alpha = P \circ T_\alpha$ it is clear that $P_\alpha \in \mathscr{P}_f(X,Y)$ and $P_\alpha(x) \to P(x)$.

Finally, since $T_\alpha(B_X) \subset B_X$ we have that for every $x \in B_X$

$$|P_\alpha(x)| = |P(T_\alpha(x))| \leq \|P\| \leq 1.$$

Note that the previous argument does not work with BAP instead of MAP.

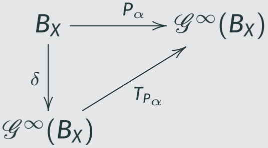
Linearization of bounded holomorphic mappings

Theorem

X has the MAP if and only if $\mathscr{G}^\infty(B_X)$ has the MAP.

Proof. ($\Leftarrow$) 1-complemented subspaces inherit the MAP.

($\Rightarrow$) For the mapping $\delta \in \overline{B}_{\mathscr{H}^\infty(B_X,\mathscr{G}^\infty(B_X))}$ the previous proposition provides us of a net $(P_\alpha) \subset \overline{B}_{\mathscr{P}_f(X,\mathscr{G}^\infty(B_X))}$ such that $P_\alpha(x) \to \delta(x)$ for all $x \in B_X$. Linearizing the polynomials we have the commutative diagram



Note that T_{P_α} are finite rank mappings with $\|T_{P_\alpha}\| = \|P_\alpha\| \leq 1$.

Also, $T_{P_\alpha}(\delta(x)) = P_\alpha(x) \to \delta(x)$, then $T_{P_\alpha} \to Id$ on $\operatorname{span} \delta(B_X)$. Since the net (T_{P_α}) is bounded the same holds for the closure. Hence, $\mathscr{G}^\infty(B_X)$ has the MAP.

It is also true that

X has the AP if and only if $\mathscr{G}^\infty(B_X)$ has the AP.

The proof is much more complicated: it is needed a net $(P_\alpha) \subset \mathscr{P}_f(X, \mathscr{G}^\infty(B_X))$ satisfying $P_\alpha \to \delta$ in a certain topology such that $T_{P_\alpha} \overset{\tau_0}{\to} Id$.

Open problem: X has the BAP $\Rightarrow \mathscr{G}^\infty(B_X)$ has the BAP?

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