Linearization of non-linear functions II

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Linearization of homogeneous polynomials

A function between Banach spaces $P : X \to Y$ is a k-homogeneous polynomial if there exists a k-linear mapping $A : X \times \cdots \times X \to Y$ such that $P(x) = A(x, \dots, x)$ for $x \in X$.

A k-homogeneous polynomial $P : X \to Y$ is continuous if and only if there is a constant $C > 0$ such that for every $x \in X$,

$$\|P(x)\| \leq C\|x\|^k.$$

For each continuous k-homogeneous polynomial $P : X \to Y$ we define a norm $\|P\|$ to be the infimum of all constants C satisfying the previous inequality. That is,

$$\|P\| = \sup\{\|P(x)\| : \ x \in B_X\}.$$

With this norm, $\mathscr{P}(^k X, Y)$ is a Banach space.

There is a linearization procedure for this space through the symmetric projective tensor product.

Linearization of homogeneous polynomials

Inside $\bigotimes^k X$ the span of all elementary symmetric tensors $\otimes^k x = x \otimes \cdots \otimes x$ is the symmetric tensor product denoted by $\bigotimes^{k,s} X$.

For $\mathbb{K} = \mathbb{C}$, we can restrict to sums instead linear combinations. Indeed, $\sum_{i=1}^n \lambda_i \otimes^k x_i = \sum_{i=1}^n \otimes^k (\lambda_i^{1/k} x_i)$ and $\sum_{i=1}^n |\lambda_i| \|x_i\|^k = \sum_{i=1}^n \|\lambda_i^{1/k} x_i\|^k$. Analogously, for $\mathbb{K} = \mathbb{R}$, we can just consider $u = \sum_{i=1}^n \epsilon_i \otimes^k x_i$ with $\epsilon_i = \pm 1$.

For the sake of simplicity, we will confine ourselves to complex scalars.

We define a norm on $\bigotimes^{k,s} X$, called the symmetric projective norm:

$$\pi_s(u) = \inf \left\{ \sum_{i=1}^n \|x_i\|^k : \ u = \sum_{i=1}^n \otimes^k x_i \right\}.$$

This is a norm and satisfies: $\pi_s(\otimes^k x) = \|x\|^k$ and $\pi_s(u) \geq \pi(u)$.

Linearization of homogeneous polynomials

We denote by $\bigotimes_{\pi_s}^{k,s} X$ the k-symmetric tensor product of X endowed with the symmetric projective norm, which is a normed space and its completion by $\widehat{\bigotimes}_{\pi_s}^{k,s} X$.

Ray Ryan in 1980 in his doctoral thesis proved that through this space there is a *linearization procedure* for $\mathscr{P}({}^k X, Y)$ with all the properties that we have described.

General linearization procedure

Consider $\mathscr{C}(U, Y) = \{f : U \to Y \text{ function of a certain class}\}$, where U is a given type of set, Y is a Banach space and $\mathscr{C}(U, Y)$ results to be a Banach space with a norm denoted by $\|\cdot\|_{\mathscr{C}}$.

Our *linearization procedure* consists in obtaining a Banach space $\mathscr{G}(U)$ and a canonical mapping $\delta : U \to \mathscr{G}(U)$ satisfying:

- For each $f \in \mathscr{C}(U, Y)$ there is a linear mapping $T_f \in \mathscr{L}(\mathscr{G}(U), Y)$ such that $f = T_f \circ \delta$.
- $\delta \in \mathscr{C}(U, \mathscr{G}(U))$ and $\|\delta\|_{\mathscr{C}} = 1$.
- The mapping

$$\begin{aligned} \mathscr{C}(U, Y) &\to \mathscr{L}(\mathscr{G}(U), Y) \\ f &\mapsto T_f \end{aligned}$$

  is a linear surjective isometry.

Linearization of homogeneous polynomials

For $\mathscr{P}({}^kX, Y) = \{P : X \to Y \text{ continuous } k\text{-homogeneous polynomial}\}$ we have the Banach space $\widehat{\bigotimes}^{k,s}_{\pi_s} X$ and the mapping $\delta : X \to \widehat{\bigotimes}^{k,s}_{\pi_s} X$ given by $\delta(x) = \otimes^k x$ satisfying

- For each $P \in \mathscr{P}({}^kX, Y)$ there is a linear mapping $T_P \in \mathscr{L}(\widehat{\bigotimes}^{k,s}_{\pi_s} X, Y)$ such that $P = T_P \circ \delta$. If $u = \sum_{i=1}^n \otimes^k x_i$ then $T_P(u) = \sum_{i=1}^n P(x_i)$ is well defined and linear. Also, for each representation of u, $\|T_P(u)\| \leq \sum_{i=1}^n \|P\| \|x_i\|^k$. Hence, $\|T_P\| \leq \|P\|$ and T_P extends to the closure with the same norm. ✓
- $\delta \in \mathscr{P}({}^kX, \widehat{\bigotimes}^{k,s}_{\pi_s} X)$ and $\|\delta\| = 1$. The mapping δ is a k-homogeneous polynomial since it is the restriction to the diagonal of the k-linear mapping

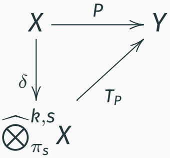
$$X \times \cdots \times X \to \widehat{\otimes}^{k,s}_{\pi_s} X$$
$$(x_1, \ldots, x_k) \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}.$$

Also, $\pi_s(\delta(x)) = \pi_s(\otimes^k x) = \|x\|^k$, so $\|\delta\| = 1$. ✓

Linearization of homogeneous polynomials

- The mapping $\mathscr{P}(^kX,Y) \to \mathscr{L}(\widehat{\bigotimes}_{\pi_s}^{k,s} X, Y)$ given by $P \mapsto T_P$ is a linear surjective isometry. It is clear that $P \mapsto T_P$ is linear. Also, for each $T \in \mathscr{L}(\widehat{\bigotimes}_{\pi_s}^{k,s} X, Y)$ we have $T \circ \delta \in \mathscr{P}(^kX,Y)$ with $\|T \circ \delta\| \leq \|T\|$. Appealing to the first bullet, the result holds. ✓

Thus, we have the commutative diagram and properties:



1. $(\widehat{\bigotimes}_{\pi_s}^{k,s} X)^* \cong \mathscr{P}(^kX)$.
2. $\operatorname{span} \delta(X)$ is dense in $\widehat{\bigotimes}_{\pi_s}^{k,s} X$.
3. $\widehat{\bigotimes}_{\pi_s}^{k,s} X$ is unique (unless isometric isomorphism).
4. If $(P_i)_i$ is a bounded net in $\mathscr{P}(^kX)$ and $P \in \mathscr{P}(^kX)$ then $T_{P_i} \overset{w^*}{\to} T_P$ if and only if $P_i(x) \to P(x)$ for every $x \in X$.

Linearization of homogeneous polynomials

Recall $\pi_s(u) \geq \pi(u)$; let us see that the inequality could be strict.

Example

Let $\{e_n\}$ be the canonical basis of ℓ_1. Then,
$u = e_1 \otimes e_2 + e_2 \otimes e_1 \in \bigotimes^{2,s} \ell_1$ satisfies $\pi(u) = 2 < 4 = \pi_s(u)$.

Clearly, $\pi(u) \leq \|e_1\|\|e_2\| + \|e_2\|\|e_1\| = 2$. Also, let
$B \in \mathrm{Bil}(\ell_1 \times \ell_1) \cong (\ell_1 \widehat{\otimes}_\pi \ell_1)^*$ given by $B(x, y) = (x_1 + x_2) \cdot (y_1 + y_2)$.
Then $\|B\| = 1$ and

$$\pi(u) \geq |T_B(u)| = |B(e_1, e_2) + B(e_2, e_1)| = 2.$$

Note that $u = \frac{\otimes^2(e_1+e_2) - \otimes^2(e_1-e_2)}{2}$ then $\pi_s(u) \leq \frac{\|e_1+e_2\|^2 + \|e_1-e_2\|^2}{2} = 4$.

Now, let $P \in \mathscr{P}(^2\ell_1) \cong (\widehat{\bigotimes}^{2,s}_{\pi_s} \ell_1)^*$ given by $P(x) = 4x_1x_2$. Then,
$\|P\| = 1$. Indeed, $|4x_1x_2| \leq (|x_1| + |x_2|)^2 \leq \|x\|^2$ and $P(\frac{e_1+e_2}{2}) = 1$.

Finally,

$$\pi_s(u) \geq |T_P(u)| = \left| \frac{P(e_1 + e_2) - P(e_1 - e_2)}{2} \right| = 4.$$

Linearization of homogeneous polynomials

For a set C in a Banach space X we denote by $\Gamma(C)$ the absolute convex hull of C; that is the set of all linear combinations $\sum_{i=1}^{n} a_i x_i$ with $x_i \in C$ for all i and $\sum_{i=1}^{n} |a_i| \leq 1$.

Consequence of Hahn-Banach theorem

If $C \subset \overline{B}_X$ is a norming set for X^* then $\overline{\Gamma}(C) = \overline{B}_X$.

Particular case

$\overline{B}_{\widehat{\bigotimes}^{k,s}_{\pi_s} X} = \overline{\Gamma}(\delta(B_X))$.

Proof. Since $\delta(B_X) \subset \overline{B}_{\widehat{\bigotimes}^{k,s}_{\pi_s} X}$ is a norming set for $\mathscr{P}(^k X)$ the result holds.

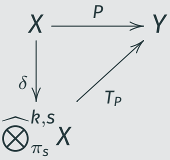
Linearization of homogeneous polynomials

A k-homogeneous polynomial $P : X \to Y$ is compact (weakly compact) if $\overline{P(B_X)}$ is a compact (weakly compact) set of Y.

Proposition

The polynomial P is compact (weakly compact) if and only if the linear operator T_P is compact (weakly compact).

Proof. Recall



Since $P(B_X) \subset T_P(B_{\widehat{\otimes}^{k,s}_{\pi_s} X})$ the implication $(\Leftarrow)$ is clear.

For $(\Rightarrow)$ we use that $\overline{\Gamma}$ respects compact and weakly compact sets and the chain of equalities

$$\overline{T_P(B_{\widehat{\otimes}^{k,s}_{\pi_s} X})} = \overline{T_P(\overline{B}_{\widehat{\otimes}^{k,s}_{\pi_s} X})} = \overline{T_P(\overline{\Gamma}(\delta(B_X)))} = \overline{\Gamma}(T_P(\delta(B_X))) = \overline{\Gamma}(P(B_X)).$$

Linearization of homogeneous polynomials

Proposition

X is isomorphic to a complemented subspace of $\widehat{\bigotimes}_{\pi_s}^{k,s} X$.

Proof. We prove it for $k = 2$. The general case follows by a similar, more involved argument which shows that $\widehat{\bigotimes}_{\pi_s}^{k,s} X$ is isomorphic to a complemented subspace of $\widehat{\bigotimes}_{\pi_s}^{(k+1),s} X$ [Blasco, Studia Math. (1997)].

Take $x_o \in X$ and $x_0^* \in X^*$ such that $x_0^*(x_0) = 1 = \|x_0\| = \|x_0^*\|$. Let $j : X \to \widehat{\bigotimes}_{\pi_s}^{2,s} X$ be given by

$$\begin{aligned} j(x) &= x \otimes x_0 + x_0 \otimes x - x_0^*(x)(\otimes^2 x_0) \\ &= \frac{\otimes^2(x + x_0) - \otimes^2(x - x_0)}{2} - x_0^*(x)(\otimes^2 x_0). \end{aligned}$$

It is clear that j is linear and $\|j\| \leq 5$.

Now, define a linear mapping $q : \widehat{\bigotimes}_{\pi_s}^{2,s} X \to X$ by $q(\otimes^2 x) = x_0^*(x)x$. Note that $\|q\| = 1$. Indeed, for $u = \sum_{i=1}^{n} \otimes^2 x_i$:

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$$\|q(u)\| = \left\| \sum_{i=1}^{n} x_0^*(x_i)x_i \right\| \leq \sum_{i=1}^{n} \|x_i\|^2.$$

Since this is valid for each representation of u we obtain that $\|q(u)\| \leq \pi_s(u)$. Also, $\|q(\otimes^2 x_0)\| = \|x_0\|^2 = \pi_s(\otimes^2 x_0)$.

Finally, we have to show that $q \circ j = Id_X$:

$$q(j(x)) = \frac{x_0^*(x+x_0)(x+x_0) - x_0^*(x-x_0)(x-x_0)}{2} - x_0^*(x)x_0^*(x_0)x_0 = x.$$

As a consequence of this result we derive that if $\widehat{\bigotimes}_{\pi_s}^{k,s} X$ is reflexive then X should be reflexive. Let us see in the following example that the reverse implication is false.

Example

$\widehat{\bigotimes}^{k,s}_{\pi_s} \ell_2$ is not reflexive.

Indeed, $\widehat{\bigotimes}^{k,s}_{\pi_s} \ell_2$ contains a copy of ℓ_1. Moreover, $\widehat{\bigotimes}^{k,s}_{\pi_s} \ell_2$ contains a complemented isometric copy of ℓ_1.

Let D be the closed subspace of $\widehat{\bigotimes}^{k,s}_{\pi_s} \ell_2$ generated by the elementary tensors $\otimes^k e_n$, where $\{e_n\}$ is the canonical basis of ℓ_2.

For $u = \sum_{n=1}^N \alpha_n \otimes^k e_n$ we have that $\pi_s(u) \leq \sum_{n=1}^N |\alpha_n|$. Consider now the polynomial $P \in \mathscr{P}(^k\ell_2)$ given by $P(x) = \sum_{n=1}^N sgn(\alpha_n)^{-1} x_n^k$, where $sgn(\alpha_n) = \frac{\alpha_n}{|\alpha_n|}$. We have

$$|P(x)| \leq \sum_{n=1}^N |x_n|^k \leq \left(\sum_{n=1}^N |x_n|^2 \right)^{k/2} = \|x\|^k \quad \text{and} \quad |P(e_n)| = 1.$$

Hence $\|P\| = 1$ and

$$\pi_s(u) \geq |T_P(u)| = \left| \sum_{n=1}^N \alpha_n P(e_n) \right| = \sum_{n=1}^N |\alpha_n|,$$

implying that D is isometric to ℓ_1.

Linearization of homogeneous polynomials

Finally, linearizing the k-homogeneous polynomial $Q : \ell_2 \to \widehat{\bigotimes}^{k,s}_{\pi_s} \ell_2$, $Q(x) = \sum_{n=1}^{\infty} x_n^k \otimes^k e_n$ we obtain $T_Q : \widehat{\bigotimes}^{k,s}_{\pi_s} \ell_2 \to \widehat{\bigotimes}^{k,s}_{\pi_s} \ell_2$ satisfying:

- $Im(T_Q) \subset D$,
- $T_Q(\otimes^k e_n) = Q(e_n) = \otimes^k e_n$, (i.e., $T_Q|_D = Id$)
- $\|T_Q\| = \|Q\| = 1$.

Thus, D is 1-complemented in $\widehat{\bigotimes}^{k,s}_{\pi_s} \ell_2$.

Let us comment without proof that sometimes the projective symmetric tensor product of a reflexive space is reflexive. For instance:

$\widehat{\bigotimes}^{k,s}_{\pi_s} \ell_p$ is reflexive for every $k < p < \infty$.