Linearization of non-linear functions I

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What this course is about?

In Banach space theory the most prominent class of functions are the continuous linear mappings.

The main reason is that they preserve the structure of the underlying Banach spaces.

But they are not the only interesting functions to study...

In this course we will deal with a linearization procedure that allows us to associate to certain non-linear functions appropriate linear mappings which retain some of the properties of the original functions. In this transaction we obtain better behaving functions, but we have to accept more complex domains in exchange.

General linearization procedure

Consider $\mathscr{C}(U,Y) = \{f : U \to Y \text{ function of a certain class}\}$, where U is a given type of set, Y is a Banach space and $\mathscr{C}(U,Y)$ results to be a Banach space with a norm denoted by $\|\cdot\|_{\mathscr{C}}$.

Our *linearization procedure* consists in obtaining a Banach space $\mathscr{G}(U)$ and a canonical mapping $\delta : U \to \mathscr{G}(U)$ satisfying:

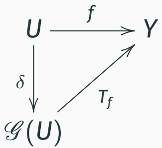
- For each $f \in \mathscr{C}(U,Y)$ there is a linear mapping $T_f \in \mathscr{L}(\mathscr{G}(U),Y)$ such that $f = T_f \circ \delta$.
- $\delta \in \mathscr{C}(U,\mathscr{G}(U))$ and $\|\delta\|_{\mathscr{C}} = 1$.
- The mapping

$$\begin{aligned} \mathscr{C}(U,Y) &\to \mathscr{L}(\mathscr{G}(U),Y) \\ f &\mapsto T_f \end{aligned}$$

  is a linear surjective isometry.

General linearization procedure

In this case we thus have the following commutative diagram



with the following properties:

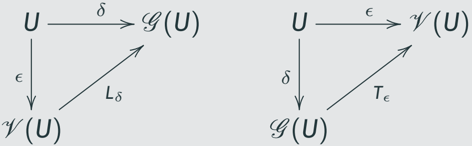
1. $\mathscr{G}(U)^* \cong \mathscr{C}(U)$.
2. $\operatorname{span} \delta(U)$ is dense in $\mathscr{G}(U)$.
3. $\mathscr{G}(U)$ is unique (unless isometric isomorphism).
4. If $(f_i)_i$ is a bounded net in $\mathscr{C}(U)$ and $f \in \mathscr{C}(U)$ then $T_{f_i} \xrightarrow{w^*} T_f$ if and only if $f_i(u) \to f(u)$ for every $u \in U$.

Proof

1. If $Y = \mathbb{C}$, the mapping $f \mapsto T_f$ is a linear surjective isometry between $\mathscr{C}(U)$ and $\mathscr{G}(U)^*$.

General linearization procedure

2. If $T_f \in \mathscr{G}(U)^*$ satisfies $T_f|_{\mathrm{span}(\delta(U))} \equiv 0$ then $f = T_f \circ \delta \equiv 0$ and hence $T_f \equiv 0$.
3. Suppose that there is another Banach space $\mathscr{V}(U)$ and a mapping $\epsilon : U \to \mathscr{V}(U)$ satisfying the same conditions. Since $\delta \in \mathscr{C}(U, \mathscr{G}(U))$ and $\epsilon \in \mathscr{C}(U, \mathscr{V}(U))$ there are commutative diagrams



Then, $L_\delta \circ T_\epsilon \circ \delta = \delta$ meaning that $L_\delta \circ T_\epsilon|_{\mathrm{span}(\delta(U))} = Id$. By 2., $L_\delta \circ T_\epsilon = Id_{\mathscr{G}(U)}$. Analogously, $T_\epsilon \circ L_\delta = Id_{\mathscr{V}(U)}$. Since $\|T_\epsilon\| = \|L_\delta\| = 1$ we derive that $\mathscr{V}(U)$ and $\mathscr{G}(U)$ are isometrically isomorphic.

General linearization procedure

4. ($\Rightarrow$) If $T_{f_i} \xrightarrow{w^*} T_f$ then $T_{f_i}(\delta(u)) \to T_f(\delta(u))$ for all $u \in U$ and so $f_i(u) \to f(u)$.
   ($\Leftarrow$) If $f_i(u) \to f(u)$ for all $u \in U$ then $T_{f_i}(v) \to T_f(v)$ for all $v \in \mathrm{span}(\delta(U))$. Also, there is a constant $C > 0$ such that $\|f_i\| \leq C$ for all i and $\|f\| \leq C$. Now, given $w \in \mathscr{G}(U)$ and $\varepsilon > 0$ there is $v \in \mathrm{span}(\delta(U))$ such that $\|w - v\| < \frac{\varepsilon}{3C}$. For this v there is i_0 such that if $i \geq i_0$, $|T_{f_i}(v) - T_f(v)| \leq \frac{\varepsilon}{3}$. Putting all together we get to

$$\begin{aligned} |T_{f_i}(w) - T_f(w)| &\leq |T_{f_i}(w) - T_{f_i}(v)| + |T_{f_i}(v) - T_f(v)| \\ &\qquad + |T_f(v) - T_f(w)| \\ &< \|T_{f_i}\|\|w - v\| + \frac{\varepsilon}{3} + \|T_f\|\|w - v\| < \varepsilon. \end{aligned}$$

General linearization procedure

We now present several cases where we can produce such a linearization procedure. In each of the settings we address some of the following questions or goals:

- If the set U is contained in or related to a Banach space X, is there exist a linear inclusion of X into $\mathscr{G}(U)$? If the answer is affirmative, is this inclusion complemented?
- If the set U is contained in or related to a Banach space X, which properties of X (separable, reflexive, metric approximation property, etc) inherits $\mathscr{G}(U)$?
- Produce a good description of the closed unit ball of $\mathscr{G}(U)$. Is it the closed absolute convex hull of $\delta(U)$?
- Identify properties of the functions in $\mathscr{C}(U, Y)$ that translate into similar properties of linear mappings T_f.

Linearization of bilinear mappings

If X, Y and Z are Banach spaces, a bilinear mapping $B : X \times Y \to Z$ is continuous if and only if there is a constant $C > 0$ such that for every $x \in X$, $y \in Y$,

$$\|B(x, y)\| \leq C\|x\|\|y\|.$$

For each continuous bilinear mapping $B : X \times Y \to Z$ we define a norm $\|B\|$ to be the infimum of all constants C satisfying the previous inequality. With this norm, $\mathrm{Bil}(X \times Y, Z)$ is a Banach space.

There is a linearization procedure for this space through the projective tensor product.

This begins with a classical algebraic scheme: for vector spaces X and Y the tensor product $X \otimes Y$ is defined as the vector span of all the elementary tensors $x \otimes y$, for $x \in X$ and $y \in Y$.

Linearization of bilinear mappings

The elementary tensor $x \otimes y$ means the element of the algebraic dual of the set of bilinear mappings given by

$$(x \otimes y)(B) = B(x, y), \qquad \text{for all } B : X \times Y \to \mathbb{K} \text{ bilinear.}$$

Note that each element $u \in X \otimes Y$ can be written in different ways as a linear combination of elementary tensors. For instance, $x \otimes y + (-x) \otimes y = 0$ because $B(x, y) + B(-x, y) = 0$ for every bilinear mapping B.

Now we want to look at this from an analytic viewpoint. For that we define a norm on $X \otimes Y$, called the projective norm:

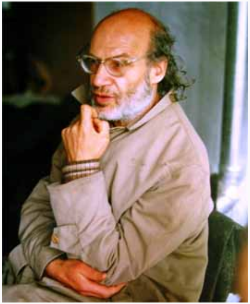
$$\pi(u) = \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| \, : \, u = \sum_{i=1}^{n} x_i \otimes y_i \right\}.$$

This is a norm and satisfies: $\pi(x \otimes y) = \|x\| \|y\|$.

Linearization of bilinear mappings

We denote by $X \otimes_\pi Y$ the tensor product of X and Y endowed with the projective norm, which is a normed space and its completion by $X \widehat{\otimes}_\pi Y$.

Alexander Grothendieck in 1953 in his *Résumé de la théorie métrique des produits tensoriels topologiques* proved that through this space there is a *linearization procedure* for $\mathrm{Bil}(X \times Y, Z)$ with all the properties that we have described.



General linearization procedure

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  is a linear surjective isometry.

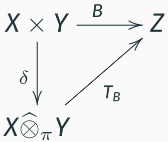
Linearization of bilinear mappings

For $\mathrm{Bil}(X \times Y, Z) = \{B : X \times Y \to Z \text{ bilinear and continuous}\}$ we have the Banach space $X\widehat{\otimes}_\pi Y$ and the mapping $\delta : X \times Y \to X\widehat{\otimes}_\pi Y$ given by $\delta(x, y) = x \otimes y$ satisfying

- For each $B \in \mathrm{Bil}(X \times Y, Z)$ there is a linear mapping $T_B \in \mathscr{L}(X\widehat{\otimes}_\pi Y, Z)$ such that $B = T_B \circ \delta$. Indeed, if $u = \sum_{i=1}^n x_i \otimes y_i$ then $T_B(u) = \sum_{i=1}^n B(x_i, y_i)$ is well defined and linear. Also, $\|T_B(u)\| \leq \sum_{i=1}^n \|B\|\|x_i\|\|y_i\|$ for each representation of u. Hence, $\|T_B(u)\| \leq \|B\|\pi(u)$, so T_B is bounded: $\|T_B\| \leq \|B\|$. ✓
- $\delta \in \mathrm{Bil}(X \times Y, X\widehat{\otimes}_\pi Y)$ and $\|\delta\| = 1$. It is clear by the definition of the tensor product that δ is bilinear. Also, $\|\delta(x, y)\| = \pi(x \otimes y) = \|x\|\|y\|$, so $\|\delta\| = 1$. ✓
- The mapping $\mathrm{Bil}(X \times Y, Z) \to \mathscr{L}(X\widehat{\otimes}_\pi Y, Z)$ given by $B \mapsto T_B$ is a linear surjective isometry. It is clear from the definition that the mapping $B \mapsto T_B$ is linear. Also, for each $T \in \mathscr{L}(X\widehat{\otimes}_\pi Y, Z)$ we have $T \circ \delta \in \mathrm{Bil}(X \times Y, Z)$ with $\|T \circ \delta\| \leq \|T\|$. Thus, appealing to the first bullet, the result holds. ✓

Linearization of bilinear mappings

Therefore, we have the following commutative diagram



and the properties:

1. $(X \widehat{\otimes}_\pi Y)^* \cong \mathrm{Bil}(X \times Y)$.
2. $\operatorname{span} \delta(X \times Y)$ is dense in $X \widehat{\otimes}_\pi Y$.
3. $X \widehat{\otimes}_\pi Y$ is unique (unless isometric isomorphism).
4. If $(B_i)_i$ is a bounded net in $\mathrm{Bil}(X \times Y)$ and $B \in \mathrm{Bil}(X \times Y)$ then $T_{B_i} \xrightarrow{w^*} T_B$ if and only if $B_i(x, y) \to B(x, y)$ for every $x \in X, y \in Y$.

Linearization of bilinear mappings

We present now a couple of results related to this linearization procedure.

Proposition

X and Y are isometric to 1-complemented subspaces of $X\widehat{\otimes}_\pi Y$.

Proof. Let $y_0 \in Y$ with $\|y_0\| = 1$ and define $\iota : X \to X\widehat{\otimes}_\pi Y$ by $\iota(x) = x \otimes y_0$. It is clear that ι is linear and isometric.

Now, take $y_0^* \in Y^*$ with $\|y_0^*\| = |y_0^*(y_0)| = 1$ and define the bilinear mapping $B : X \times Y \to X$ by $B(x, y) = x \cdot y_0^*(y)$. Then, $\|B(x, y)\| \leq \|x\|\|y\|$ and $\|B(x, y_0)\| = \|x\|$ meaning that $\|B\| = 1$.

Linearizing B we get to $T_B : X\widehat{\otimes}_\pi Y \to X$ which satisfies $\|T_B\| = 1$ and $T_B \circ \iota = Id_X$. This proves the result for X. The proof for Y is analogous.

Linearization of bilinear mappings

Recall that a Banach space X has the bounded approximation property (BAP) if there exists a number $C > 0$ and a net of finite rank operators (T_α) such that $\|T_\alpha\| \leq C$ and $T_\alpha(x) \to x$ for all $x \in X$.

Proposition

X and Y have the BAP if and only if $X\widehat{\otimes}_\pi Y$ has the BAP.

Proof. $(\Leftarrow)$ is clear since complemented subspaces inherit the BAP.

$(\Rightarrow)$ Let (T_α) and (S_β) be nets of finite rank operators approximating Id_X and Id_Y, respectively, with $\|T_\alpha\| \leq C_X$ and $\|S_\beta\| \leq C_Y$. Let us consider the net $(T_\alpha \otimes S_\beta)_{(\alpha,\beta)}$ where the index set (α, β) is ordered canonically and $T_\alpha \otimes S_\beta : X\widehat{\otimes}_\pi Y \to X\widehat{\otimes}_\pi Y$ is the linearization of $[(x, y) \in X \times Y \mapsto T_\alpha(x) \otimes S_\beta(y) \in X\widehat{\otimes}_\pi Y]$. It is clear that $T_\alpha \otimes S_\beta$ are finite rank mappings and $\|T_\alpha \otimes S_\beta\| \leq C_X \cdot C_Y$.

Linearization of bilinear mappings

To see that they approximate the identity, it is enough to check their values in elementary tensors:

$$\pi\left((T_\alpha \otimes S_\beta)(x \otimes y) - x \otimes y\right) = \pi\left((T_\alpha(x) - x) \otimes S_\beta(y) + x \otimes (S_\beta(y) - y)\right)$$
$$\leq \|T_\alpha(x) - x\| C_Y \|y\| + \|x\| \|S_\beta(y) - y\|.$$

Hence, $\pi\left((T_\alpha \otimes S_\beta)(x \otimes y) - x \otimes y\right) \to 0$.

Other simple things:

- X and Y are separable if and only if $X \widehat{\otimes}_\pi Y$. One implication follows from the fact that $\operatorname{span} \delta(X \times Y)$ is dense in $X \widehat{\otimes}_\pi Y$. The other implication is clear from the contention of X and Y inside the tensor product.

Linearization of bilinear mappings

- The image of B is contained in a finite dimensional subspace (separable subspace) if and only if the range of T_B is finite dimensional (separable). Note that the image of B is contained in the image of T_B. And the other way around again is consequence of the denseness of $\operatorname{span} \delta(X \times Y)$ in the tensor product.
- If B is surjective then so is T_B. The reverse implication is false. The first sentence is obvious. For the second, take the bilinear mapping $\delta : X \times Y \to X \widehat{\otimes}_\pi Y$ which is not surjective and note that T_δ is the identity mapping $Id : X \widehat{\otimes}_\pi Y \to X \widehat{\otimes}_\pi Y$.