Symplectic structures on Rochberg spaces

Lluís Santaló School: Linear and non-linear analysis in Banach spaces

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Phillips [1940] The exact sequence

$0 \longrightarrow c_0 \overset{i}{\longrightarrow} \ell_\infty \overset{q}{\longrightarrow} \ell_\infty/c_0 \longrightarrow 0$ is not trivial

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Another solution to this problem was given by Kalton and Peck [1979]

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where

- $\Omega : Z \to Y$ is homogeneous and there is $K > 0$ such that for all $z_1, z_2 \in Z$,

$$\|\Omega(z_1 + z_2) - \Omega(z_1) - \Omega(z_2)\| \leq K(\|z_1\| + \|z_2\|).$$

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Ω is trivial if it may be written as $\Omega = B + L$.

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Kalton-Peck [1979] The quotient map $Z_2 \to \ell_2$ is strictly singular.

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Remark.

1. $L_\omega^*(x) = -L_\omega(x)$, for every $x \in X \subseteq X^{**}$.
2. Every symplectic Banach space is reflexive.

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Equivalence

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Two symplectic spaces (X_1, ω_1) *and* (X_2, ω_2) *are equivalent if there is an isomorphism* $T : X_1 \to X_2$ *such that* $\omega_2(Tx, Ty) = \omega_1(x, y)$.

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Kalton-Swanson [1982] (Z_2, ω) is not trivial.

Complex method of interpolation

Riesz-Thorin: Let $T : \ell_\infty \to \ell_\infty$ be a bounded operator which is also bounded as a map from ℓ_1 to ℓ_1. Then for every $p \in (1, \infty)$ we have that T is bounded from ℓ_p to ℓ_p and

$$\|T : \ell_p \to \ell_p\| \leq \|T : \ell_\infty \to \ell_\infty\|^{1-\frac{1}{p}} \|T : \ell_1 \to \ell_1\|^{\frac{1}{p}}$$

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$$X_\theta = \{f(\theta) \,:\, f \in \mathcal{F}\}, \|x\|_{X_\theta} = \inf\{\|f\|_{\mathcal{F}} \,:\, f \in \mathcal{F},\, f(\theta) = x\}$$

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Example The Kalton-Peck space Z_2 can be obtained as a derived space: $\ell_2 = (\ell_\infty, \ell_1)_{1/2}$,

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Definition. Rochberg [1996]

$\mathfrak{R}_\theta^{(n)} = \{(x_{n-1}, ..., x_1, x_0) \in \ell_\infty^n : x_i = \frac{f^{(i)}(\theta)}{i!}$, for some $f \in \mathcal{F}$, and all $0 \leq i \leq n-1\}$ equipped with the canonical quotient norm.

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where $i(x_{n-1}, \ldots, x_0) = (x_{n-1}, \ldots, x_0, 0, \ldots, 0)$ and $\pi(x_{m+n-1}, \ldots, x_{k-1}, \ldots, x_0) = (x_{k-1}, \ldots, x_0)$.

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Cabello - Castillo - Correa [2019] $\omega_n : \mathfrak{R}^{(n)} \times \mathfrak{R}^{(n)} \to \mathbb{R}$ given by

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For the odd case, let $\sigma(x) = (-x_2, x_1, -x_4, x_3, \dots)$ on ℓ_2. The induced diagonal operator τ_σ is bounded on $\mathfrak{R}^{(n)}$.

$\overline{\omega_n}\big((x_{n-1}, \dots, x_0), (y_{n-1}, \dots, y_0)\big) :=$
$\omega_n\big((x_{n-1}, \dots, x_0), \tau_\sigma(y_{n-1}, \dots, y_0)\big) = \sum_{i+j=n-1} (-1)^i \langle x_i, \sigma y_j \rangle.$

Operators on Rochberg spaces

Proposition. An operator $\tau : \mathfrak{R}^{(n)} \to X$ either is strictly singular or there exists a complemented subspace E of $\mathfrak{R}^{(n)}$ with $E \simeq \mathfrak{R}^{(n)}$ such that $\tau_{|E}$ is an isomorphism.

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The proof follows arguments of Kalton [82] and Castillo -Correa- Ferenczi- González [21]

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- $\tau : \mathfrak{R}^{(n)} \to X$ is $s.s \iff \tau_{|\ell_2}$ is $s.s$

Operators on Rochberg spaces

Proposition. An operator $\tau : \mathfrak{R}^{(n)} \to X$ either is strictly singular or there exists a complemented subspace E of $\mathfrak{R}^{(n)}$ with $E \simeq \mathfrak{R}^{(n)}$ such that $\tau_{|E}$ is an isomorphism.

The proof follows arguments of Kalton [82] and Castillo -Correa-Ferenczi- González [21]

- Cabello - Castillo - Kalton [2015] The exact sequence $0 \longrightarrow \ell_2 \xrightarrow{\ i\ } \mathfrak{R}^{(n)} \xrightarrow{\ \pi\ } \mathfrak{R}^{(n-1)} \longrightarrow 0$ has singular quotient map.
- $\tau : \mathfrak{R}^{(n)} \to X$ is $s.s \iff \tau_{|\ell_2}$ is $s.s$
- Let τ be a non-$s.s$ operator. WLOG we can assume that $\tau_{|\ell_2}$ is an embedding.

Operators on Rochberg spaces

$$\begin{array}{ccccc}
\ell_2 & & & & \\
\Big\downarrow{\scriptstyle i} & & & & \\
\mathfrak{R}^{(n)} & \xrightarrow{(\tau,\mathbf{id})} & X \oplus \mathfrak{R}^{(n)} & \longrightarrow & X \\
\Big\downarrow{\scriptstyle \pi} & & & & \\
\mathfrak{R}^{(n-1)} & & & &
\end{array}$$

Operators on Rochberg spaces

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\mathfrak{R}^{(n-1)} & \longrightarrow & PO & \longrightarrow & X
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\end{array}
$$

- $Q(\tau, \mathbf{id})$ is $s.s$ since it factors through π.
- $Q(\tau, \mathbf{id}) = Q(\tau, 0) + Q(0, \mathbf{id})$.
- $Q(0, \mathbf{id})$ is an embedding.
- $Q(\tau, 0)$ is a upper semi-Fredholm.
- τ is an isomorphism on some finite codimensional subspace of $\mathfrak{R}^{(n)}$.

Theorem [CCGP] Let $T \in \mathcal{L}(\mathfrak{R}^{(n)})$. If $T^{+}T$ is strictly singular then T is strictly singular.

Symplectic structures on Rochberg spaces

Theorem [CCGP] Let $T \in \mathcal{L}(\mathfrak{R}^{(n)})$. If T^+T is strictly singular then T is strictly singular.

The symplectic dual T^+ is defined as

$$\omega_n(T^+x, y) = \omega_n(x, Ty)$$

Symplectic structures on Rochberg spaces

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$$\omega_n(T^+x, y) = \omega_n(x, Ty)$$

Theorem [CCGP] $\mathfrak{R}^{(n)}$ is symplectic non-trivial for every $n > 1$.

The end

Gracias!