Symplectic structures on Rochberg spaces

Lluís Santaló School: Linear and non-linear analysis in Banach spaces

Wilson A. Cuéllar (Universidade de São Paulo) Joint work with J. Castillo, M. González and R. Pino

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Phillips [1940] The exact sequence

 $0 \longrightarrow c_0 \xrightarrow{i} \ell_{\infty} \xrightarrow{q} \ell_{\infty}/c_0 \longrightarrow 0 \text{ is not trivial}$

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Another solution to this problem was given by Kalton and Peck [1979]

Kalton-Peck theory

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where

• $\Omega: Z \to Y$ is homogeneous and there is K > 0 such that for all $z_1, z_2 \in Z$,

$$\|\Omega(z_1+z_2) - \Omega(z_1) - \Omega(z_2)\| \le K(\|z_1\| + \|z_2\|).$$

• $||(y,z)||_{\Omega} = ||y - \Omega z||_{Y} + ||z||_{Z}$

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 Ω is trivial if it may be written as $\Omega = B + L$.

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Kalton-Peck [1979] The quotient map $Z_2 \rightarrow \ell_2$ is strictly singular.

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Remark.

- 1. $L^*_{\omega}(x) = -L_{\omega}(x)$, for every $x \in X \subseteq X^{**}$.
- 2. Every symplectic Banach space is reflexive.

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3. $X = \ell_2$ with $\omega(x, y) = \langle x, \sigma y \rangle$, where
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5. Kalton-Peck space Z_2

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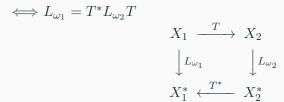
$$\omega((x,y), (x',y')) = \langle x, y' \rangle - \langle y, x' \rangle$$

Equivalence

Definition

Two symplectic spaces (X_1, ω_1) and (X_2, ω_2) are equivalent if there is an isomorphism $T: X_1 \to X_2$ such that

$$\omega_2(Tx,Ty) = \omega_1(x,y).$$

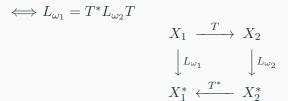


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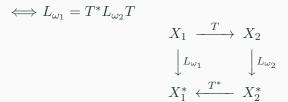
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Weinstein [1971] Every symplectic structure on a Hilbert space is trivial.

Kalton-Swanson [1982] (Z_2, ω) is not trivial.

Complex method of interpolation

Riesz-Thorin: Let $T: \ell_{\infty} \to \ell_{\infty}$ be a bounded operator which is also bounded as a map from ℓ_1 to ℓ_1 . Then for every $p \in (1, \infty)$ we have that T is bounded from ℓ_p to ℓ_p and

$$||T: \ell_p \to \ell_p|| \le ||T: \ell_\infty \to \ell_\infty||^{1-\frac{1}{p}} ||T: \ell_1 \to \ell_1||^{\frac{1}{p}}$$

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$$\begin{split} \|T:\ell_p\to\ell_p\|\leq \|T:\ell_\infty\to\ell_\infty\|^{1-\frac{1}{p}}\|T:\ell_1\to\ell_1\|^{\frac{1}{p}} \\ \text{Notation.} \ S=\{z\in\mathbb{C}\,:\,0<\Re(z)<1\} \end{split}$$

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1.
$$f$$
 is analytic on S
2. $f(it) \in \ell_{\infty}$ and $f(1+it) \in \ell_1$ for every $t \in \mathbb{R}$.
3. $t \mapsto f(j+it)$ is continuous and bounded $(j = 0, 1)$
 $||f|| = \max \left\{ \sup_{t \in \mathbb{R}} ||f(it)||_{\ell_{\infty}}, \sup_{t \in \mathbb{R}} ||f(1+it)||_{\ell_1} \right\} < \infty$

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 $X_{\theta} = \{f(\theta) : f \in \mathcal{F}\}, \|x\|_{X_{\theta}} = \inf\{\|f\|_{\mathcal{F}} : f \in \mathcal{F}, f(\theta) = x\}$

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Rochberg and Weiss [1983] There is a short exact sequence

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Example The Kalton-Peck space Z_2 can be obtained as a derived space: $\ell_2 = (\ell_{\infty}, \ell_1)_{1/2}$,

 $Z_2 = d\ell_2$

Proposition. The map $\delta_{\theta}^n : \mathcal{F} \to \ell_{\infty}$, evaluation of the *n*-th derivate at θ , is bounded for all $0 < \theta < 1$ and all $n \in \mathbb{N}$.

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Definition. Rochberg [1996]

$$\label{eq:stars} \begin{split} \mathfrak{R}_{\theta}^{(n)} &= \{(x_{n-1},...,x_1,x_0) \in \ell_{\infty}^n \colon x_i = \frac{f^{(i)}(\theta)}{i!}, \text{for some } f \in \\ \mathcal{F}, \text{and all } 0 \leq i \leq n-1 \} \text{ equipped with the canonical quotient norm.} \end{split}$$

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where $i(x_{n-1}, \ldots, x_0) = (x_{n-1}, \ldots, x_0, 0, \ldots, 0)$ and $\pi(x_{m+n-1}, \ldots, x_{k-1}, \ldots, x_0) = (x_{k-1}, \ldots, x_0).$ Notation. $\mathfrak{R}^{(n)} := \mathfrak{R}^{(n)}_{\frac{1}{2}}$

Proposition [CCGP] $\mathfrak{R}^{(n)}$ is symplectic for every $n \geq 1$.

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Cabello - Castillo - Correa [2019] $\omega_n:\mathfrak{R}^{(n)} imes\mathfrak{R}^{(n)} o\mathbb{R}$ given by

$$\omega_n((x_{n-1},\ldots,x_0),(y_{n-1},\ldots,y_0)) = \sum_{i+j=n-1} (-1)^i \langle x_i, y_j \rangle.$$

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For the odd case, let $\sigma(x) = (-x_2, x_1, -x_4, x_3, ...)$ on ℓ_2 . The induced diagonal operator τ_{σ} is bounded on $\Re^{(n)}$.

$$\overline{\omega_n}\big((x_{n-1},\ldots,x_0),(y_{n-1},\ldots,y_0)\big) := \\ \omega_n\big((x_{n-1},\ldots,x_0),\tau_\sigma(y_{n-1},\ldots,y_0)\big) = \sum_{i+j=n-1}(-1)^i \langle x_i,\sigma y_j \rangle.$$

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• Cabello - Castillo - Kalton [2015] The exact sequence $0 \longrightarrow \ell_2 \xrightarrow{i} \mathfrak{R}^{(n)} \xrightarrow{\pi} \mathfrak{R}^{(n-1)} \longrightarrow 0$ has singular quotient map. **Proposition.** An operator $\tau : \mathfrak{R}^{(n)} \to X$ either is strictly singular or there exists a complemented subspace E of $\mathfrak{R}^{(n)}$ with $E \simeq \mathfrak{R}^{(n)}$ such that $\tau_{|E}$ is an isomorphism.

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$$\tau : \mathfrak{R}^{(n)} \to X \text{ is } s.s \iff \tau_{|\ell_2} \text{ is } s.s$$

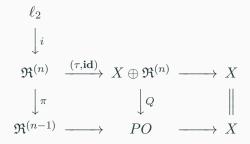
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• Cabello - Castillo - Kalton [2015] The exact sequence

- $\tau: \mathfrak{R}^{(n)} \to X \text{ is } s.s \iff \tau_{|\ell_2} \text{ is } s.s$
- Let τ be a non-s.s operator. WLOG we can assume that $\tau_{|\ell_2}$ is an embedding.

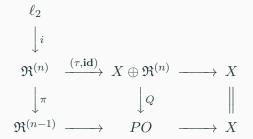
$$\begin{array}{c} \ell_2 \\ \downarrow^i \\ \mathfrak{R}^{(n)} \xrightarrow{(\tau, \mathbf{id})} X \oplus \mathfrak{R}^{(n)} \longrightarrow X \\ \downarrow^{\pi} \\ \mathfrak{R}^{(n-1)} \end{array}$$



Operators on Rochberg spaces

$$\begin{array}{c} \ell_2 \\ \downarrow^i \\ \mathfrak{R}^{(n)} \xrightarrow{(\tau, \mathbf{id})} X \oplus \mathfrak{R}^{(n)} \longrightarrow X \\ \downarrow^{\pi} & \downarrow^Q & \parallel \\ \mathfrak{R}^{(n-1)} \longrightarrow PO \longrightarrow X \end{array}$$

Operators on Rochberg spaces



- $Q(\tau, id)$ is s.s since it factors through π .
- $Q(\tau, id) = Q(\tau, 0) + Q(0, id).$
- $Q(0, \mathbf{id})$ is an embedding.
- $Q(\tau, 0)$ is a upper semi-Fredholm.
- τ is an isomorphism on some finite codimensional subspace of $\Re^{(n)}$.

Theorem [CCGP] Let $T \in \mathcal{L}(\mathfrak{R}^{(n)})$. If T^+T is strictly singular then T is strictly singular.

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Theorem [CCGP] $\Re^{(n)}$ is symplectic non-trivial for every n > 1.

Gracias!