On the numerical index of 2-dimensional Lipschitz-free spaces

Based on a joint work with Antonio José Guirao and Vicente Montesinos

Christian Cobollo

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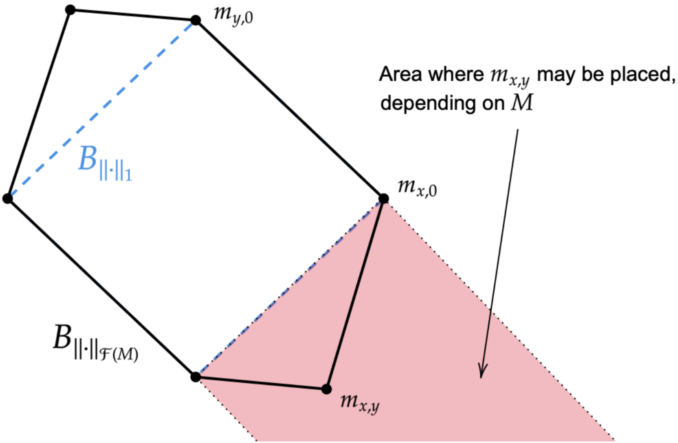
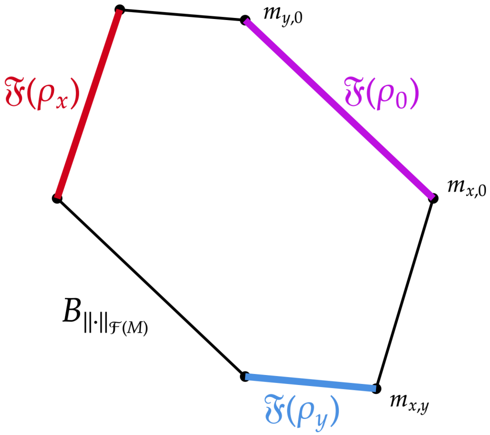
Introduction

- Lipschitz-free spaces
- Numerical index



$m_{y,0}$
$B_{\|\cdot\|_1}$
$m_{x,0}$
Area where $m_{x,y}$ may be placed, depending on M
$B_{\|\cdot\|_{\mathcal{F}(M)}}$
$m_{x,y}$



$m_{y,0}$
$\mathfrak{F}(\rho_x)$
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$m_{x,0}$
$B_{\|\cdot\|_{\mathcal{F}(M)}}$
$m_{x,y}$
$\mathfrak{F}(\rho_y)$

$T \in \mathcal{L}(X)$, its **numerical radius** is defined as

$$\nu(T) = \sup\{|\langle x^*, Tx\rangle| : x \in S_X, x^* \in S_{X^*}, \langle x, x^*\rangle = 1\}$$

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Let $M = (\{x,y,z\}, d)$, $T \in \mathcal{L}(\mathcal{F}(M))$, then

$$\nu(T) = \max\{\nu(T, m_{x,y}), \nu(T, m_{x,z}), \nu(T, m_{y,z})\}.$$

$n(\mathcal{F}(M))$ Preliminaries

The **numerical index** of X is defined as

$$n(X) := \inf\{\nu(T) : T \in S_{\mathcal{L}(X)}\}$$

Goal: find $T \in S_{\mathcal{L}(X)}$ minimizing

$$\max\{\nu(T, m_{x,y}), \nu(T, m_{x,z}), \nu(T, m_{y,z})\}$$

Gromov product

$$G_z(x,y) := d(x,z) + d(y,z) - d(x,y)$$

The first lower bound, the optimal contribution

Some metric tools

Gromov product

$$G_z(x, y) := d(x, z) + d(y, z) - d(x, y)$$

Weighted Gromov product

$$\gamma_z(x, y) := d(x, y)G_z(x, y)$$

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Optimal Contribution

$$\nu_{\mathsf{op}}(x,y) := \frac{\gamma_z(x,y)}{\gamma_y(x,z) + \gamma_x(y,z)}$$

The first lower bound, the optimal contribution

Lemma

Let $M = (\{x, y, z\}, d)$ be a triangle, and $T \in S_{\mathcal{F}(M)}$ be such that $\|Tm_{x,y}\| = 1$. Then,
$$\nu(T, m_{x,y}) \geq \nu_{\mathsf{op}}(x, y).$$

The first lower bound, the optimal contribution

Lemma

Let $M = (\{x, y, z\}, d)$ be a triangle, and $T \in S_{\mathcal{F}(M)}$ be such that $\|Tm_{x,y}\| = 1$. Then,

$$\nu(T, m_{x,y}) \geq \nu_{\text{op}}(x, y).$$

Moreover, if

$$Tm_{x,y} = \lambda_z m_{x,z} + (1 - \lambda_z) m_{y,z},$$

where

$$\lambda_z := \frac{\gamma_y(x, z)}{\gamma_y(x, z) + \gamma_x(y, z)},$$

then $\nu(T, m_{x,y}) = \nu_{\text{op}}(x, y)$.

The first lower bound, the optimal contribution

Proposition

Let $M = (\{x, y, z\}, d)$ be a metric space. Then, $n(\mathcal{F}(M)) \in [\frac{1}{2}, 1]$.
Moreover, if M is not an equilateral triangle, then $n(\mathcal{F}(M)) > 1/2$.

The second lower bound, the metric ratio

Lemma

Let $M = (\{x, y, z\}, d)$ be a triangle, and $T \in S_{\mathcal{F}(M)}$ be such that $\|Tm_{x,y}\| = 1$. Then,

$$\nu(T) \geq R_z(x, y).$$

Combining the bounds

If $Tm_{x,y} \in S_{\mathcal{F}(M)}$, then

$$\nu(T) \geq \nu_{\mathsf{op}}(x,y)$$
$$\nu(T) \geq R_z(x,y)$$

Corollary

Let $M = (\{x,y,z\}, d)$ be a triangle and $T \in S_{\mathcal{L}(\mathcal{F}(M))}$ satisfying $\|Tm_{x,y}\| = 1$. Then,

$$\nu(T) \geq \max\{\nu_{\mathsf{op}}(x,y), R_z(x,y)\}.$$

Combining the bounds

Proposition

Let $M = (\{x, y, z\}, d)$ a triangle. Then

$$n(\mathcal{F}(M)) \geq \min \Big\{ \max\{\nu_{\mathsf{op}}(x,y), R_z(x,y)\}, \\ \max\{\nu_{\mathsf{op}}(x,z), R_y(x,z)\}, \\ \max\{\nu_{\mathsf{op}}(y,z), R_x(y,z)\} \Big\}$$

Combining the bounds

Proposition

Let $M = (\{x, y, 0\}, d)$ be a triangle with $d(x, y) \geq d(x, 0) \geq d(y, 0)$. Then,

$$n(\mathcal{F}(M)) \geq \max\{\nu_{\mathsf{op}}(x, 0), R_y(x, 0)\}.$$

Combining the bounds

Theorem

Let $M = (\{x, y, 0\}, d)$ be a triangle with $d(x, y) \geq d(x, 0) \geq d(y, 0)$. Then,

$$n(\mathcal{F}(M)) = \max\{\nu_{\mathsf{op}}(x, 0), R_y(x, 0)\}.$$

The formula

Theorem

Let $M = (\{x, y, 0\}, d)$ be a metric space with $d(x, y) \geq d(x, 0) \geq d(y, 0)$. Then:

- if M is aligned, then $n(\mathcal{F}(M)) = 1$;
- otherwise, if M is a triangle, then

$$n(\mathcal{F}(M)) = \max\left\{\nu_{\mathsf{op}}(x, 0), R_y(x, 0)\right\}.$$

Corollary

Let $M = (\{x, y, 0\}, d)$ be an isosceles triangle such that $d(x, y) = d(x, 0) \geq d(y, 0)$. Then

$$n(\mathcal{F}(M)) = \frac{d(x, 0)}{d(x, 0) + d(y, 0)}.$$

In particular, $n(\mathcal{F}(M)) \in [\frac{1}{2}, 1)$.

Some Aplications

Hexagonal norms

Corollary

Let $M = (\{x, y, 0\}, d)$ be an isosceles triangle such that $d(x, y) \geq d(x, 0) = d(y, 0)$. Then

$$n(\mathcal{F}(M)) = \frac{d(x,0)}{3d(x,0) - d(x,y)}.$$

In particular, $n(\mathcal{F}(M)) \in [\frac{1}{2}, 1)$.

M. Martín and J. Merí, *Numerical index of some polyhedral norms on the plane* **(2007)**

Some Aplications

Infinite dimensional spaces

We might even use triangles in some infinite-dimensional constructions:

Corollary

We can construct infinite dimensional spaces $\mathcal{F}(M)$ with $n(\mathcal{F}(M)) = \alpha$ for every $\alpha \in [\frac{1}{2}, 1]$

$$n(\mathcal{F}(\bigoplus_1 M_i)) = n(\bigoplus_1 \mathcal{F}(M_i)) = \inf\{n(\mathcal{F}(M_i))\} = \alpha$$

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Theorem

Let $A \subset (\mathbb{R}^n, \|\cdot\|_2)$, $n \geq 2$, with non-empty interior. Then, $\mathcal{F}(A)$ is a separable infinite-dimensional Lipschitz-free space such that, for every $\alpha \in [\frac{1}{2}, 1]$, it contains a 2-dimensional subspace Y_α with $n(Y_\alpha) = \alpha$.

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The End

Thanks For Your Attention!