Banach spaces and Banach lattices

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$|x| = x \vee -x$

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- Spaces with unconditional basis with coordinatewise order: ℓ_2, ℓ_p....

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$$\text{Example: } \left|\sum_{i=1}^{n} \lambda_i x_i\right| \leq \sqrt{\sum_{i=1}^{n} \lambda_i^2} \cdot \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

Krivine version of Grothendieck inequality

Theorem

If $T : X \longrightarrow Y$ is a linear operator between Banach lattices, then

$$\left\| \sqrt{\sum_{i=1}^{n} |Tx_i|^2} \right\| \leq K_G \cdot \|T\| \cdot \left\| \sqrt{\sum_{i=1}^{n} |x_i|^2} \right\|$$

The free Banach lattice generated by a Banach space E

For $x \in E$, take $\delta_x : E^* \longrightarrow \mathbb{R}$ the evaluation.

Theorem (A., Tradacete, Rodríguez)

The free Banach lattice generated by E is the closure of the the vector lattice generated by $\{\delta_e : e \in E\}$ in $\mathbb{R}^{E^*}$ under the norm

$$\|f\| = \sup\left\{\sum_{i=1}^{m} |f(x_i^*)| : \sup_{x \in B_E} \sum_{i=1}^{m} |x_i^*(x)| \leq 1\right\}$$

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So whenever we have an ℓ_3-basis (x_n) in a Banach lattice, the absolute values $(|x_n|)$ satisfy an upper $\ell_{6/5}$-estimate.

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For the reverse inequality, we use Walsh matrices.

Weakly compactly generated

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- $FBL[\ell_2(\Gamma)]$ is not weakly compactly generated, because the absolute values of the basis generate $\ell_1(\Gamma)$.
- $FBL[c_0]$ and $FBL[\ell_p(\Gamma)]$ are weakly compactly generated if $p > 2$.
- Therefore, the Banach lattice generated by any Banach space copy of $c_0(\Gamma)$ or $\ell_p(\Gamma)$ is WCG if $p > 2$.

Open questions

- Characterize the bases whose absolute values in $FBL[E]$ are weakly null.
- Characterize the Banach spaces E for which $FBL[E]$ is WCG.