The Schur and Radon-Nikodým properties in Lipschitz-free spaces

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(joint work with C. Gartland, C. Petitjean & A. Procházka)

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- ... **Radon-Nikodým property (RNP)** if $\overline{\mathrm{conv}}\,\mathrm{dent}\,C = C$ for every closed bounded convex $C \subset X$

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- Both are hereditary and invariant under isomorphisms
- ℓ_1 has both properties, L_1 has neither
- In general, neither property implies the other one

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Question (implicit in Kalton 2004)

For which Lipschitz-free Banach spaces do they hold?

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Question (implicit in Kalton 2004)

For which Lipschitz-free Banach spaces do they hold?

Answer (Aliaga, Gartland, Petitjean, Procházka 2022)

Both properties are equivalent for Lipschitz-free spaces.

Lipschitz spaces

Let (M, d) be a complete metric space. Fix a base point $0 \in M$.
The *Lipschitz constant* of $f : M \to \mathbb{R}$ is

$$\|f\|_L := \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x \neq y \in M \right\}$$

The *Lipschitz space*

$$\mathrm{Lip}_0(M) = \{f : M \to \mathbb{R} : \|f\|_L < \infty, f(0) = 0\}$$

is a Banach space with norm $\|\cdot\|_L$.

Lipschitz-free spaces

For $x \in M$, consider the evaluation operators

$$\delta(x) : f \mapsto f(x)$$

Then $\delta : M \to \mathrm{Lip}_0(M)^*$ is a (nonlinear) isometric embedding.

Lipschitz-free space (or Arens-Eells space)

$$\mathcal{F}(M) = \overline{\mathrm{span}}\,\delta(M) \subset \mathrm{Lip}_0(M)^*$$

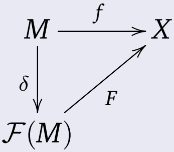
Theorem (Arens, Eells 1956)

$$\mathcal{F}(M)^* \equiv \mathrm{Lip}_0(M)$$

Universal property

Theorem (Kadets 1985; Pestov 1986; Weaver 1999)

Let M be a metric space, X be a Banach space.
Let $f\colon M \to X$ be a **Lipschitz mapping** such that $f(0) = 0$.
Then there is a **linear operator** $F\colon \mathcal{F}(M) \to X$ with $F|_{\delta(M)} = f$ and $\|F\| = \|f\|_L$.



In other words, $\mathrm{Lip}_0(M,X) \equiv \mathcal{B}(\mathcal{F}(M),X)$.

Universal property

Theorem (Godefroy, Kalton 2003)

Let M, N be metric spaces.
Let $f\colon M \to N$ be a **Lipschitz mapping** such that $f(0) = 0$.
Then there is a **linear operator** $F\colon \mathcal{F}(M) \to \mathcal{F}(N)$ with $F|_{\delta(M)} = f$ and $\|F\| = \|f\|_L$.

$$\begin{array}{ccc} M & \xrightarrow{\;f\;} & N \\ {\scriptstyle \delta_M}\downarrow & & \downarrow{\scriptstyle \delta_N} \\ \mathcal{F}(M) & \xrightarrow[\;F\;]{} & \mathcal{F}(N) \end{array}$$

Lipschitz-free subspaces

In particular:

- If M and N are bi-Lipschitz equivalent then $\mathcal{F}(M) \sim \mathcal{F}(N)$.
- If $N \subset M$ then $\mathcal{F}(N) \subset \mathcal{F}(M)$ isometrically.

Lipschitz-free subspaces

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- If $N \subset M$ then $\mathcal{F}(N) \subset \mathcal{F}(M)$ isometrically.

Theorem (Aliaga, Pernecká 2020)

Let $S_i \subset M$ be closed subsets. Then

$$\bigcap_i \mathcal{F}(S_i) = \mathcal{F}\left(\bigcap_i S_i\right)$$

Examples

- $\mathcal{F}(\mathbb{N}) \equiv \ell_1$:

$$\begin{aligned} T\colon \mathcal{F}(\mathbb{N}) &\to \ell_1 \\ \delta(n) &\mapsto (1, \overset{n}{\ldots}, 1, 0, 0, \ldots) \end{aligned}$$

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- $\mathcal{F}([0,1]) \equiv L_1[0,1]$:

$$\begin{aligned} T\colon \mathcal{F}([0,1]) &\to L_1[0,1] \\ \delta(x) &\mapsto \chi_{[0,x]} \end{aligned}$$

- $\mathcal{F}(\mathbb{R}) \equiv L_1(\mathbb{R})$ similarly

Schur and RNP in $\mathcal{F}(M)$: The compact case

Locally flat functions

A function f is **locally flat** at x if $\lim_{r\to 0} \left\| f|_{B(x,r)} \right\|_L = 0$.

We denote $\mathrm{lip}_0(M)$ = locally flat functions in $\mathrm{Lip}_0(M)$.

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- $M \subset \mathbb{R}$ and $\lambda(M) = 0$:

Fix $\varepsilon > 0$ and let $U \subset \mathbb{R}$ open, $U \supset M$, $\lambda(U) < \varepsilon$. Let $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \lambda([0,x] \setminus U) = \int_0^x (1 - \chi_U(t))\, dt$$

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Then f is locally constant on $M \Longrightarrow f|_M \in \mathrm{lip}_0(M)$.
Moreover $\|f\|_L = 1$ and $|f(y) - f(x)| > |y - x| - \varepsilon$.

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- $M = \mathbb{N}$: every f is locally constant $\implies$ locally flat
- $M \subset \mathbb{R}$ and $\lambda(M) = 0$:

Uniform separation

We say that $\mathrm{lip}_0(M)$ **SPU** (*separates points of M uniformly*) if

$$\forall x,y \in M, \varepsilon > 0 \text{ there is } f \in \mathrm{lip}_0(M) \text{ such that } \|f\|_L \leq 1 \text{ and}$$
$$f(x) - f(y) > d(x,y) - \varepsilon$$

Uniform separation and duality

Theorem (Weaver 1996)

If K is compact and $\mathrm{lip}_0(K)$ SPU then $\mathrm{lip}_0(K)^* = \mathcal{F}(K)$.

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Examples:

- K has a snowflake metric (Johnson 1970)
- K is countable (Dalet 2015; Weaver 2018)
- K is ultrametric (Dalet 2015)
- $K \subset$ a $\mathbb{R}$-tree, $\lambda(K) = 0$ (Aliaga, Petitjean, Procházka 2021)
- $\mathcal{H}^1(K) = 0$

Schur & RNP: Compact case

$$\mathrm{lip}_0(K) \text{ SPU}$$

$$\Updownarrow$$

$$\mathrm{lip}_0(K)^* = \mathcal{F}(K)$$

$$\Downarrow$$

$$\mathcal{F}(K) \text{ dual}$$

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Separable dual spaces have the RNP.

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$$\mathrm{lip}_0(K)^* = \mathcal{F}(K) \Longrightarrow \mathcal{F}(K) \text{ Schur}$$

$$\Downarrow$$

$$\mathcal{F}(K) \text{ dual} \Longrightarrow \mathcal{F}(K) \text{ RNP}$$

Theorem (Kalton 2004; Petitjean 2017)

If $\mathrm{lip}_0(M)$ is 1-norming then $\mathcal{F}(M)$ has the Schur property.

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$$\mathrm{lip}_0(K) \text{ SPU}$$

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$$\mathrm{lip}_0(K)^* = \mathcal{F}(K) \Longrightarrow \mathcal{F}(K) \text{ Schur}$$

$$\Downarrow$$

$$\mathcal{F}(K) \text{ dual} \Longrightarrow \mathcal{F}(K) \text{ RNP} \Longrightarrow L_1 \not\hookrightarrow \mathcal{F}(K)$$

($\mathcal{F}(K)$ Schur $\Longrightarrow L_1 \not\hookrightarrow \mathcal{F}(K)$)

L_1 is neither Schur nor RNP.

Purely 1-unrectifiable spaces

A **curve fragment** is a bi-Lipschitz copy of a compact $K \subset \mathbb{R}$ with positive measure.
A metric space M is **purely 1-unrectifiable** if it contains no curve fragments.

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A metric space M is **purely 1-unrectifiable** if it contains no curve fragments.

Theorem (Godard 2010)

Let $M \subset \mathbb{R}$ closed and infinite.

- If $\lambda(M) = 0$ then $\mathcal{F}(M) \equiv \ell_1$.
- If $\lambda(M) > 0$ then $\mathcal{F}(M) \sim L_1$.

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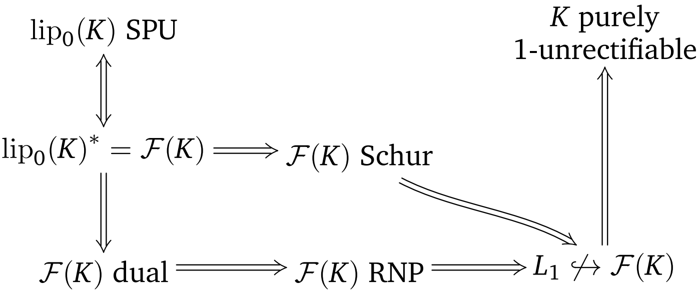
Corollary

If M contains a curve fragment, then $L_1 \hookrightarrow \mathcal{F}(M)$.

Schur & RNP: Compact case



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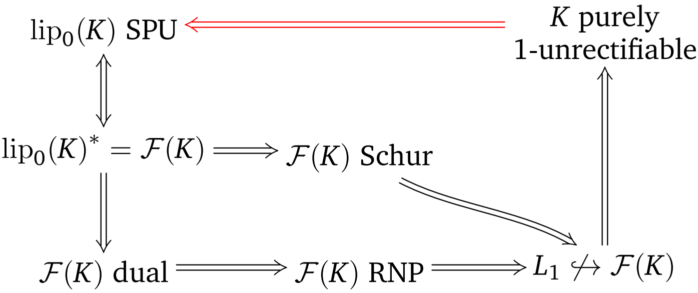

Theorem (Aliaga, Gartland, Petitjean, Procházka 2022)

If K is compact and purely 1-unrectifiable then $\mathrm{lip}_0(K)$ SPU.

Argument using Bate's results

Let K be compact and purely 1-unrectifiable, and $f \in B_{\mathrm{Lip}_0(K)}$.

Lemma (Bate 2020)

For any $\varepsilon > 0$ there are $g \in B_{\mathrm{Lip}_0(K)}$ and $\delta > 0$ such that:

1. $\|f - g\|_\infty \leq \varepsilon$
2. $\dfrac{|g(x) - g(y)|}{d(x,y)} \leq \dfrac{|f(x) - f(y)|}{d(x,y)} + \varepsilon$ for all $x \neq y \in K$
3. $\dfrac{|g(x) - g(y)|}{d(x,y)} \leq \varepsilon$ for all $x \neq y \in K$ with $d(x,y) \leq \delta$

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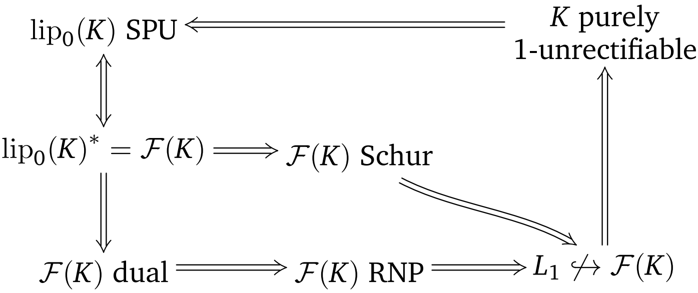
3. $\dfrac{|g(x) - g(y)|}{d(x,y)} \leq \varepsilon$ for all $x \neq y \in K$ with $d(x,y) \leq \delta$

Theorem

For any $\varepsilon > 0$ there is $g \in B_{\mathrm{lip}_0(K)}$ such that $\|f - g\|_\infty \leq \varepsilon$.

Proof: Fix $\sum \varepsilon_n \leq \varepsilon$ and apply Bate's lemma iteratively to get a sequence (g_n) where $\|g_n - g_{n-1}\| \leq \varepsilon_n$. Then $g_n \xrightarrow{\|\cdot\|_\infty} g \in \mathrm{lip}_0(K)$.

Schur & RNP: Compact case



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Theorem (Aliaga, Gartland, Petitjean, Procházka 2022)

For a compact metric space K, TFAE:

- K is purely 1-unrectifiable
- $\mathrm{lip}_0(K)$ separates points of K uniformly
- $\mathrm{lip}_0(K)^* = \mathcal{F}(K)$
- $\mathcal{F}(K)$ is a dual Banach space
- $\mathcal{F}(K)$ has the Schur property
- $\mathcal{F}(K)$ has the Radon-Nikodým property
- $\mathcal{F}(K)$ does not contain L_1

Schur and RNP in $\mathcal{F}(M)$: The general case

Kalton's lemma

Let M be any complete metric space.
For $E \subset M$, use the notation

$$[E]_r = \bigcup_{e \in E} B(e, r)$$

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If $W = (w_n) \subset \mathcal{F}(M)$ is a weakly null sequence, then

- for every $\varepsilon, r > 0$ there is a finite $E \subset M$ such that

$$W \subset \mathcal{F}([E]_r) + \varepsilon B_{\mathcal{F}(M)}$$

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i.e. there is a sequence $(v_n) \subset \mathcal{F}([E]_r)$ s.t. $\|w_n - v_n\| \leq \varepsilon$

Tightness

Proposition (Aliaga, Noûs, Petitjean, Procházka 2021)

For a subset $W \subset \mathcal{F}(M)$, TFAE:

- For every $\varepsilon, r > 0$ there is a finite $E \subset M$ such that

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- For every $\varepsilon > 0$ there is a compact $K \subset M$ such that
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Moreover there is a linear (non-continuous) map $T : W \to \mathcal{F}(K)$ such that $\|w - Tw\| \leq \varepsilon$ for $w \in W$.
(In fact T is a uniform limit of operators on W.)

Tightness

Sketch of proof:

Let $r_n \searrow 0$ and iteratively approximate W in

- $\mathcal{F}([E_1]_{r_1})$
- $\mathcal{F}([E_1]_{r_1}) \cap \mathcal{F}([E_2]_{r_2})$
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The approximations converge to something contained in

$$\bigcap_{n=1}^{\infty} \mathcal{F}([E_n]_{r_n}) = \mathcal{F}\left(\bigcap_{n=1}^{\infty} [E_n]_{r_n}\right)$$

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Compact determination

Theorem (Aliaga, Noûs, Petitjean, Procházka 2021)

$\mathcal{F}(M)$ is Schur iff $\mathcal{F}(K)$ is Schur for every compact $K \subset M$.

We say that the Schur property is **compactly determined**.

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Sketch of proof ($\Leftarrow$):

Let $w_n \xrightarrow{w} 0$ in $\mathcal{F}(M)$.
Find $(v_n) \subset \mathcal{F}(K)$ with $v_n \approx w_n$.
Using the linearity of $w_n \mapsto v_n$, we have $v_n \xrightarrow{w} 0$.
By assumption we get $v_n \to 0$.
Thus $w_n \to 0$ as well.

Compact determination

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Other compactly determined properties, with similar proof:

- weak sequential completeness
- the approximation property (AP)
- the Dunford-Pettis property

...and with a probabilistic version of the proof:

- the Radon-Nikodým property

Theorem (Aliaga, Gartland, Petitjean, Procházka 2022)

$\mathcal{F}(M)$ is RNP iff $\mathcal{F}(K)$ is RNP for every compact $K \subset M$.

Probabilistic Kalton's lemma

Fix a probability measure space (Ω, Σ, μ).

A Banach space X has the RNP iff every uniformly integrable X-valued martingale converges in $L_1(X) = L_1(\Omega, \Sigma, \mu; X)$.

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Lemma

Every martingale $W = (w_n) \subset L_1(\mathcal{F}(M))$ satisfies

- for every $\varepsilon, r > 0$ there is a finite $E \subset M$ such that

$$W \subset L_1(\mathcal{F}([E]_r)) + \varepsilon B_{L_1(\mathcal{F}(M))}$$

Mean tightness

Proposition

For a subset $W \subset L_1(\mathcal{F}(M))$, TFAE:

- For every $\varepsilon, r > 0$ there is a finite $E \subset M$ such that

$$W \subset L_1(\mathcal{F}([E]_r)) + \varepsilon B_{L_1(\mathcal{F}(M))}$$

- For every $\varepsilon > 0$ there is a compact $K \subset M$ such that

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Moreover there is a linear map $T : W \to L_1(\mathcal{F}(K))$ such that $\|w - Tw\|_{L_1} \leq \varepsilon$ for $w \in W$.

The general case: summary

$$\begin{array}{c} \mathcal{F}(K) \text{ RNP} \\ \text{for compact } K \subset M \\ \Updownarrow \\ \mathcal{F}(M) \text{ RNP} \end{array} \qquad \begin{array}{c} \mathcal{F}(K) \text{ Schur} \\ \text{for compact } K \subset M \\ \Updownarrow \\ \mathcal{F}(M) \text{ Schur} \end{array}$$

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$\Updownarrow$ (diagonal, to the left column) $\quad$ $\Updownarrow$ (vertical, to the right column)

$\mathcal{F}(K)$ RNP for compact $K \subset M$	$\mathcal{F}(K)$ Schur for compact $K \subset M$
$\Updownarrow$	$\Updownarrow$
$\mathcal{F}(M)$ RNP	$\mathcal{F}(M)$ Schur

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all $K \subset M$ purely 1-unrectifiable $\Longleftrightarrow$ M purely 1-unrectifiable

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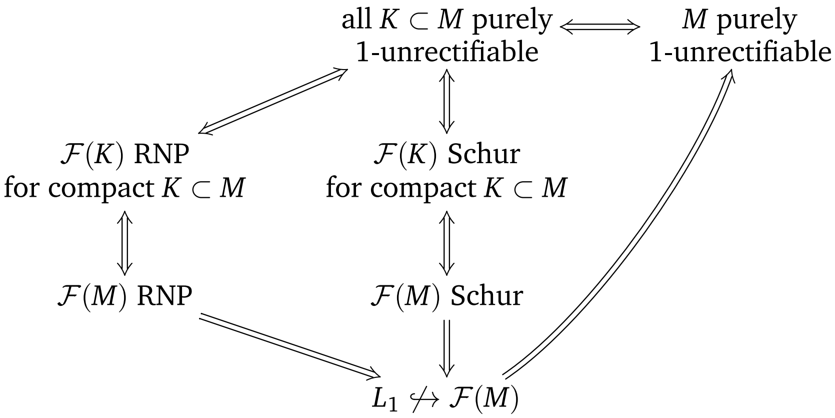

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For a complete metric space M, TFAE:

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For a complete metric space M, TFAE:

- M is purely 1-unrectifiable
- $\mathcal{F}(M)$ does not contain L_1
- $\mathcal{F}(M)$ has the Schur property
- $\mathcal{F}(M)$ has the Radon-Nikodým property
- $\mathcal{F}(M)$ has the Krein-Milman property

X has the **Krein-Milman property (KMP)** if $\overline{\mathrm{conv}}\,\mathrm{ext}\, C = C$ for every closed bounded convex $C \subset X$.

$$\text{RNP} \implies \text{KMP} \implies L_1 \not\hookrightarrow X$$

Duality in the general case

What about duality for general M?

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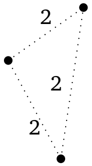
NO. Counterexample:

(García-Lirola, Petitjean, Procházka, Rueda Zoca 2018)



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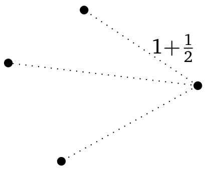

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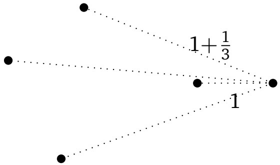

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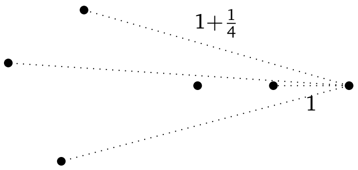

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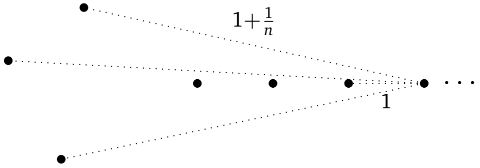

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Separable M: **YES.**

Duality in the general case

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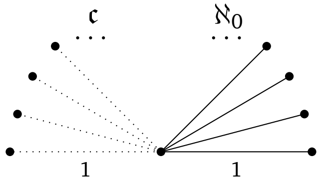
Separable M: **YES.**

Nonseparable M: **NO.** Counterexample: $\mathcal{F}(M) = C([0,1])^*$

(Aliaga, Procházka)



Open question

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Does Schur/RNP imply the AP for Lipschitz-free spaces?

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Equivalently:

Open question (Godefroy)

If K compact and $\mathrm{lip}_0(K)$ SPU, does $\mathcal{F}(K)$ have the (M)AP?

Thank you for your attention!

References:

- R. J. Aliaga, C. Gartland, C. Petitjean and A. Procházka, **"Purely 1-unrectifiable metric spaces and locally flat Lipschitz functions"**, *Trans. Amer. Math. Soc.* 375, pp. 3529–3567, 2022
- R. J. Aliaga, C. Noûs, C. Petitjean and A. Procházka, **"Compact reduction in Lipschitz-free spaces"**, *Studia Math.* 260, pp. 341–359, 2021
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