Lipschitz-free spaces and representing measures

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Let M be a complete metric space with base point 0, and let $\mathrm{Lip}_0(M)$ denote the Banach space of Lipschitz functions on M that vanish at 0. The *Lipschitz-free* space $\mathcal{F}(M) \subseteq \mathrm{Lip}_0(M)^*$ over M is defined as the closed linear span of the set of functionals $f \mapsto f(x)$, $x \in M$, and is an isometric predual of $\mathrm{Lip}_0(M)$. These spaces have important applications in the linear and nonlinear theory of Banach spaces.

Let $\widetilde{M} = \{(x,y) \in M \times M : x \neq y\}$ and denote by $\beta\widetilde{M}$ its Stone-Čech compactification. Define the *de Leeuw transform* $\Phi : \mathrm{Lip}_0(M) \to C(\beta\widetilde{M})$ by

$$\Phi f(x,y) = \frac{f(x) - f(y)}{d(x,y)}, \quad f \in \mathrm{Lip}_0(M),\ (x,y) \in \widetilde{M},$$

and extending continuously to $\beta\widetilde{M}$. This is a linear isometric embedding whose dual $\Phi^* : C(\beta\widetilde{M})^* \to \mathrm{Lip}_0(M)^*$ is therefore a quotient map. Thus, every element of $\mathrm{Lip}_0(M)^*$ can be represented (non-uniquely) by Radon measures on $\beta\widetilde{M}$ via the de Leeuw transform.

Given $\phi \in \mathrm{Lip}_0(M)^*$, there exists a positive Radon measure μ on $\beta\widetilde{M}$ such that $\Phi^*\mu = \phi$ and $\|\mu\| = \|\phi\|$. We call such a measure μ an *optimal representation* of ϕ. In this mini-course, we investigate the properties of optimal representations μ such that $\mu(\widetilde{M}) = \|\mu\|$ (in which case $\Phi^*\mu \in \mathcal{F}(M)$) or more generally $\mu(\widetilde{M}) > 0$. We illustrate connections with cyclical monotonicity from optimal transport theory, elements of $\mathcal{F}(M)$ that can be induced by measures on M, and present an application of this work to the extreme point problem in Lipschitz-free spaces.

This is joint work with Ramón Aliaga (Universitat Politècnica de València) and Eva Pernecká (Czech Technical University, Prague).